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Pricing Variable Annuity Guarantees in a local volatility framework

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HIGHLIGHTS

- We study the price of Variable Annuity Guarantees (GAO and GMIB).
- We use a local volatility model with stochastic interest rates.
- We present a method to calibrate the local volatility model.
- We compare prices obtained in a local, stochastic and constant volatility framework.

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ABSTRACT

In this paper, we study the price of Variable Annuity Guarantees, particularly those of Guaranteed Annuity Options (GAO) and Guaranteed Minimum Income Benefit (GMIB), in the settings of a derivative pricing model where the underlying spot (the fund) is locally governed by a geometric Brownian motion with local volatility, while interest rates follow a Hull–White one-factor Gaussian model. Notwithstanding the fact that in this framework, the local volatility depends on a particularly complex expectation where no closed-form expression exists and it is neither directly related to European call prices or other liquid products, we present in this contribution a method based on Monte Carlo Simulations to calibrate the local volatility model. We further compare the Variable Annuity Guarantee prices obtained in three different settings, namely the local volatility, the stochastic volatility and the constant volatility models all combined with stochastic interest rates and show that an appropriate volatility modeling is important for these long-dated derivatives. More precisely, we compare the prices of GAO, GMIB Rider and barrier types GAO obtained by using the local volatility, stochastic volatility and constant volatility models.

1. Introduction

Variable Annuities are the insurance contracts that propose a guaranteed return at retirement often higher than the current market rate and therefore they have become a part of many retirement plans. Variable Annuity products are typically based on an investment in a mutual fund composed of stocks and bonds (see for example Gao, 2010 and Pelsser and Schrager, 2004) and offer a range of options to give minimum guarantees and protect against negative equity movements. One of the most popular type of Variable Annuity Guarantees in Japan and North America is the Guaranteed Minimum Income Benefit (GMIB). At her retirement date, a GMIB policyholder will have the right to choose between the fund value at that time or (life) annuity payments based on the initial fund value at a fixed guarantee rate. Similar products are available in Europe under the name Guaranteed Annuity Options (GAO). Many authors have previously studied the pricing and hedging of GMIBs and GAOs assuming a geometric Brownian motion and a constant volatility for the fund value (see for example Milevsky and Promislow, 2001; Boyle and Hardy, 2003, Ballotta and Haberman, 2003, Pelsser, 2003, Biffis and Millossovich, 2006, Marshall et al., 2010, Chu and Kwok, 2007).

GAO and GMIB can be considered as long-dated options since their maturity is based on the retirement date. When pricing long-dated derivatives, it is highly recommended that the pricing model used to evaluate and hedge the products takes into account the stochastic behavior of the interest rates as well as the stochastic behavior of the fund. Furthermore, the volatility of the fund can have a significant impact and should not be neglected. It has been shown in Boyle and Hardy (2003) that the value of the fund as well as the interest rates and the mortality assumptions strongly influence the cost of these guarantees. Some authors consider the evolution of mortality stochastic as well (see for example Ballotta and Haberman, 2003 and Biffis and Millossovich, 2006). In van Haastrecht et al. (2010), van Haastrecht et al. have studied the impact of the volatility of the fund on the price of GAO by using a stochastic volatility approach.

Another category of models capable of fitting the vanilla market implied volatilities are local volatility models introduced in 1994 by Derman and Kani and by Dupire in resp. Derman and Kani

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(1994) and Dupire (1994) and recently extended by, among others, Atlan (2006) in a two-factor local volatility model with stochastic interest rates; and then by Piterbarg (2006) and Deelstra and Rayée (2012) in a three-factor model for the pricing of long-dated FX derivatives. The main advantage of local volatility models is that the volatility is a deterministic function of the equity spot and time, which avoids the issue in working in incomplete markets in comparison with stochastic volatility models. Therefore local volatility models are more appropriate for hedging strategies. The local volatility function is expressed in terms of implied volatilities or market call prices and the calibration is undertaken on the entire implied volatility surface directly. Consequently, the local volatility models usually capture in a more precise manner the surface of implied volatilities compared to the stochastic volatility models.

Stochastic volatility models are advantageous in that it is possible to derive closed-form solutions for some European derivatives. In van Haastrecht et al. (2010), the authors have derived closed-form formulae for GAO prices in the Schöbel and Zhu stochastic volatility model combined with Hull and White stochastic interest rates. However, the GMIB Rider, one of the popular products traded by insurance companies in North America (see AnnuityFYI, 2009) has a more complicated payoff compared to a pure GAO and therefore no closed-form solution exists for the price of a GMIB Rider, not even in the Schöbel and Zhu stochastic volatility model. The only way to evaluate a GMIB Rider is through the use of numerical approaches, such as Monte Carlo simulations.

In this paper, we study the prices of GAO, GMIB Riders and barrier type GAOs in the settings of a two-factor pricing model where the equity (fund) is locally governed by a geometric Brownian motion with a local volatility, while interest rates follow a Hull–White one-factor Gaussian model. In this framework, the local volatility expression contains an expectation for which no closed-form expression exists and which is unfortunately not directly related to European call prices or other liquid products. Its calculation can be done by numerical integration methods or Monte Carlo simulations. Alternative approaches are to calibrate the local volatility from stochastic volatility models by using links between local and stochastic volatility models or by adjusting the tractable local volatility surface coming from a deterministic interest rates framework (see Deelstra and Rayée, 2012).

Furthermore, we compare Variable Annuity Guarantee prices obtained in three distinct settings, namely, the local volatility, the stochastic volatility and the constant volatility models, all in the settings of stochastic interest rates. We show that using a non constant volatility for the volatility of the equity fund value can have significant impact on the value of these Variable Annuity Guarantees and that the impact generated by a local volatility model is not equivalent to the one generated by a stochastic volatility model, even if both are calibrated to the same market data.

This paper is organized as follows: Section 2 is a summary of properties of the local volatility model in a constant interest rates framework and of its extension in a stochastic interest rates framework. In Section 3, we present an approach based on Monte Carlo simulations for the calibration of the local volatility function in the stochastic interest rates setting. In Section 4, we present the three types of Variable Annuity Guarantees we study in this paper, namely, the GAO, the GMIB Rider and barrier type GAOs. In Section 4.1 we present the GAO, and then in Section 4.2 we define a GMIB Rider and finally in Section 4.3, we study two types of barrier GAO. Section 5 is devoted to numerical results. In Section 5.1, we present the calibration procedure for the Hull and White parameters and the calibration of the local volatility with respect to the vanilla market. Sections 5.2–5.4 investigate how the local volatility model behaves when pricing GAO, GMIB Rider and barrier type GAOs (respectively) with respect to the Schöbel–Zhu Hull–White stochastic volatility model and the Black–Scholes Hull–White model. Conclusions are given in Section 6.

2. The local volatility model: from a constant to a stochastic interest rates framework

In a constant interest rates setting, the risk neutral probability density of an underlying asset S can be derived from the market prices of European options. More precisely, the risk neutral price of a European Call with strike K and maturity T is given by

$$\begin{aligned} C(K,T) &= e^{-rT}\mathbf{E}^{Q}[(S_T-K)^+] \\ &= e^{-rT}\int_0^{+\infty}(x-K)^+\phi(x,T)dx \end{aligned} \tag{1}$$

where Q denotes the risk neutral measure, r is the constant interest rate and where $\phi(x,T)$ corresponds to the risk neutral probability density of the underlying asset S at time T. Differentiating this Eq. (1) twice with respect to K one obtains the well-known equality

$$\frac{\partial^2 C(K,T)}{\partial K^2} = e^{-rT}\phi(K,T).$$

Using these results, Dupire (1994) and Derman and Kani (1994) introduced in 1994, in the setting of constant interest rates, the so-called local volatility models for the underlying assets which have a deterministic time and state-dependent volatility function, consistent with the current European option prices. In a local volatility model with constant interest rate, the underlying asset S (paying a constant dividend yield q) is assumed to be governed by the following risk neutral dynamics

$$dS(t) = (r-q)S(t)dt + \sigma(t,S(t))S(t)dW_S^Q(t), \tag{2}$$

where $W_S^Q(t)$ is a Brownian motion under the risk neutral measure Q and where the diffusion function $\sigma(t,S(t))$ satisfies conditions such that Eq. (2) has a unique solution.

Dupire (1994) and Derman and Kani (1994) noted that there is a unique volatility function consistent with European option prices, and called it the local volatility function. Indeed, given a complete set of European option prices for all strikes and expirations, this local volatility function is given uniquely by

$$\sigma(T,K) = \sqrt{\frac{\frac{\partial C(K,T)}{\partial T} + (r-q)K\frac{\partial C(K,T)}{\partial K} + qC(K,T)}{\frac{1}{2}K^2\frac{\partial^2 C(K,T)}{\partial K^2}}}. \tag{3}$$

This local volatility model is very tractable since the local volatility surface can directly be computed from vanilla option market prices.

Another useful property of the local volatility model is its link with stochastic volatility models. More precisely, if the underlying spot is governed by the following risk neutral dynamics with constant interest rate,

$$dS(t) = (r-q)S(t)dt + \gamma(t,\nu(t))S(t)dW_S^Q(t), \tag{4}$$

and applying Gyöngy's mimicking theorem Gyöngy (1986), one can show that the local volatility is given by the square root of the conditional expectation under the risk neutral measure[1] Q of the instantaneous equity stochastic spot volatility to the square at the future time t, conditional on the equity spot level $S(t)$ being equal to K:

$$\sigma(t,K) = \sqrt{\mathbf{E}^{Q}[\gamma^2(t,\nu(t)) \mid S(t)=K]}. \tag{5}$$

Common designs for the function $\gamma(t,\nu(t))$ are $\nu(t)$, $\exp(\sqrt{\nu(t)})$ and $\sqrt{\nu(t)}$. The stochastic variable $\nu(t)$ is often modeled by a Cox–Ingersoll–Ross (CIR) process (such as the Heston model

[1] Assuming that the risk neutral probability measure Q used in the stochastic volatility framework is the same as the one used in the local volatility framework.