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Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting

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HIGHLIGHTS

- We investigate the two-dimensional reinsurance policy in a dynamic setting.
- By using martingale central limit theorem, a 2-dimensional diffusion is derived.
- 2-dimensional excess-of-loss reinsurance is optimal to minimize ruin probability.
- The optimal policy and the minimized ruin probability are obtained in closed form.
- The results are illustrated by numerical examples.

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ABSTRACT

Assume that an insurer has two dependent lines of business. The reserves of the two lines of business are modeled by a two-dimensional compound Poisson risk process or a common shock model. To protect from large losses and to reduce the ruin probability of the insurer, the insurer applies a reinsurance policy to each line of business, thus the two policies form a two-dimensional reinsurance policy. In this paper, we investigate the two-dimensional reinsurance policy in a dynamic setting. By using the martingale central limit theorem, we first derive a two-dimensional diffusion approximation to the two-dimensional compound Poisson reserve risk process. We then formulate the total reserve of the insurer by a controlled diffusion process and reduce the problem of optimal reinsurance strategies to a dynamic control problem for the controlled diffusion process. Under this setting, we show that a two-dimensional excess-of-loss reinsurance policy is an optimal form that minimizes the ruin probability of the controlled diffusion process. By solving a HJB equation with two dependent controls, we derive the explicit expressions of the optimal two-dimensional retention levels of the optimal two-dimensional excess-of-loss reinsurance policy and the minimized ruin probability. The results show that optimal dynamic two-dimensional retention levels are constant and the optimal retention levels are related by a deterministic function. We also illustrate the results by numerical examples.

1. Introduction

Finding optimal reinsurance strategies for insurers is an interesting research topic in risk theory. Usually, optimal reinsurance strategies can be studied either in a static setting or in a dynamic setting. The study in dynamic cases has become a popular research topic because the optimal reinsurance strategy in a dynamic setting reflects the change of an insurer's best risk position with respect to time. So far, optimal reinsurance strategies with independent risks in a dynamic setting have been extensively studied. See, for instance, Asmussen et al. (2000), Bäerle (2004), Zhang et al. (2007), Liang and Guo (2007), Bai and Guo (2008), Zhang and Siu (2009), Wei et al. (2010), Liang et al. (2011), Meng and Siu (2011), Gu et al. (2012), and references therein. However, there are few research works which consider optimal reinsurance strategies with dependent risks in a dynamic setting due to the difficulty of such a problem. In addition, some ruin-related problems with dependent risks have been studied in the literature. For example, Cossette and Marceau (2000) used a discrete-time approach to study how the common shock affects the finite-time ruin probability and the adjustment coefficient; Yuen et al. (2002) studied the ruin probability of a common shock dependent model in which the common shock follows an Erlang(2) process; and Yuen et al. (2006) considered a bivariate compound Poisson model for two dependent classes of

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insurance business and studied the ruin probability that at least one class of business will get ruined.

In this paper, we consider an insurance portfolio which has two dependent lines of business, such as auto insurance/third party insurance, casualty insurance/health insurance, life insurance/endowment insurance, and so on. The aggregated claims up to time t in the two lines of business are denoted by a bivariate compound random process $\left(\sum_{i=1}^{M_1(t)} X_i,\ \sum_{i=1}^{M_2(t)} Y_i\right)$, and the accumulated premiums up to time t in the two lines of business are denoted by $c_1(t)$ and $c_2(t)$, respectively. Thus, the total reserve of the insurance portfolio up to time t, denoted by $\{R(t), t \geq 0\}$, is

$$R(t) = x + c_1(t) + c_2(t) - \sum_{i=1}^{M_1(t)} X_i - \sum_{i=1}^{M_2(t)} Y_i, \quad t \geq 0,$$

where $x \geq 0$ is the initial reserve of the insurance portfolio.

There are different dependent risk models which describe the dependence structure of the bivariate compound random process $\left(\sum_{i=1}^{M_1(t)} X_i,\ \sum_{i=1}^{M_2(t)} Y_i\right)$. See, for example, Chan et al. (2003), Wang and Yuen (2005), and references therein. In this paper, we consider a bivariate compound Poisson risk process or a common shock model as in Yuen et al. (2006) for continuous-time models and Centeno (2005) for a static setting or a single period model. We assume that $M_1(t) = N_1(t) + N(t)$ and $M_2(t) = N_2(t) + N(t)$, where $\{N_1(t)\}$, $\{N_2(t)\}$, and $\{N(t)\}$ are independent Poisson processes. In this case, the dependence of the two lines of business is due to a common shock governed by the Poisson process $\{N(t)\}$.

Furthermore, to protect from potential large losses in the portfolio, the insurance company applies reinsurance strategies f_1 and f_2 to lines 1 and 2, respectively. Thus, the total reserve of the insurance portfolio up to time t under the reinsurance strategies f_1 and f_2, denoted by $\{R^\pi(t), t \geq 0\}$, is given by

$$R^\pi(t) = x + [c_1(t) - p_1^\pi(t)] + [c_2(t) - p_2^\pi(t)] - \sum_{i=1}^{M_1(t)} f_1(X_i) - \sum_{i=1}^{M_2(t)} f_2(Y_i), \quad t \geq 0,$$

where $\pi = (f_1(\cdot), f_2(\cdot))$ is a two-dimensional reinsurance policy with $0 \leq f_i(x) \leq x$ for $x \geq 0$ and $i = 1, 2$, and $p_1^\pi(t)$ and $p_2^\pi(t)$ are the accumulated reinsurance premiums up to time t paid to reinsurers for lines 1 and 2 corresponding to the reinsurance policy π, respectively. The goal of the insurer is to find the optimal reinsurance policy $\pi^* = (f_1^*, f_2^*)$, which minimizes the ruin probability of the insurance portfolio, namely, to find the optimal reinsurance policy $\pi^* = (f_1^*, f_2^*)$ such that

$$\Pr\{R^{\pi^*}(t) < 0 \text{ for some } t > 0\} = \min_\pi \Pr\{R^\pi(t) < 0 \text{ for some } t > 0\}.$$

For such a dependent risk model with a common shock, the explicit solutions for the ruin probability $\Pr\{R^\pi(t) < 0 \text{ for some } t > 0\}$ are not available. In order to make problems tractable and to obtain explicit solutions, we first give a diffusion approximation to the dependent risk model, and then study an optimal risk control problem for the approximated diffusion model in a dynamic setting.

The rest of this paper is organized as follows. In Section 2, by employing the martingale central limit theorem, we obtain a two-dimensional diffusion approximation to the two-dimensional reserve risk processes of the two lines of business. In the setting of the diffusion approximation, we show that an excess-of-loss reinsurance is the optimal form that minimizes the ruin probability of the diffusion reserve risk process. In Section 3, we formulate the controlled diffusion problem with excess-of-loss reinsurance policies in a dynamic setting. In Section 4, the explicit solutions to the value function or the minimized ruin probability and the optimal risk retention levels are obtained by solving a HJB equation with two dependent controls. At last, numerical results and conclusions are given in Section 5.

2. The approximated diffusion risk model

With the bivariate compound Poisson process or the common shock model and the expected value premium principle, we model the reserve risk process for the insurer with two lines of business by the following risk model

$$R(t) = x + (c_1 + c_2)t - \sum_{i=1}^{N_1(t)+N(t)} X_i - \sum_{i=1}^{N_2(t)+N(t)} Y_i, \tag{2.1}$$

where $\{N_1(t)\}$, $\{N_2(t)\}$ and $\{N(t)\}$ are three independent Poisson processes with intensities $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda > 0$, respectively. The claims sizes $\{X_i,\ i = 1, 2, \ldots\}(\{Y_i,\ i = 1, 2, \ldots\})$ are i.i.d. positive random variables with common distribution function $F_X(x)$ $(F_Y(y))$, mean value $\mu_X(\mu_Y)$, and variance σ_X (σ_Y). It is assumed that $0 = F_X(0) < F_X(x) < 1$ for $x > 0$ $(0 = F_Y(0) < F_Y(y) < 1$ for $y > 0)$. Also assume that $\{X_i\}$ and $\{Y_i\}$ are independent and both of them are independent of the three Poisson processes.

In addition, c_1 and c_2, the premium rates in lines 1 and 2 of business, respectively, are assumed to be determined by the expected value principle with positive safety loading $\eta_i,\ i = 1, 2$, namely,

$$c_1 = (1 + \eta_1)(\lambda_1 + \lambda)\mu_X,$$
$$c_2 = (1 + \eta_2)(\lambda_2 + \lambda)\mu_Y.$$

For a reinsurance policy $\pi = (f_1, f_2)$, assume the reinsurance premium also is calculated by the expected value principle with positive safety loading $\theta_i,\ i = 1, 2$, then the reserve process of the insurance portfolio with reinsurance policy π is given by

$$R^\pi(t) = x + (c_1^\pi + c_2^\pi)t - \sum_{i=1}^{N_1(t)+N(t)} f_1(X_i) - \sum_{i=1}^{N_2(t)+N(t)} f_2(Y_i), \tag{2.2}$$

where c_i^π is the net premium rate for the insurer in line i with the reinsurance strategy π, $i = 1, 2$, and c_1^π and c_2^π are given by

$$c_1^\pi = (\lambda_1 + \lambda)[(1 + \theta_1)E[f_1(X_i)] - (\theta_1 - \eta_1)\mu_X],$$
$$c_2^\pi = (\lambda_2 + \lambda)[(1 + \theta_2)E[f_2(Y_i)] - (\theta_2 - \eta_2)\mu_Y].$$

It is reasonable to assume that the reinsurance is non-cheap, that is $\theta_i > \eta_i,\ i = 1, 2$.

Now we recall diffusion approximations of stochastic processes. Let "$S_n \Longrightarrow S$" denote the stochastic process $\{S_n(t)\}$ weakly converges to the process $\{S(t)\}$ with respect to Skorohod topology as $n \to \infty$. For a risk process $\{C(t)\}$, if there exists a sequence of risk processes $\{Q_n(t),\ n = 1, 2, \ldots\}$ such that, for any $t \geq 0$,

$$Q_1(t) = C(t) \quad \text{and} \quad Q_n \Longrightarrow Q$$

with $Q(t) = C(0) + \alpha t + \beta B_t$ for some constants α, β and a standard Brownian motion $\{B_t\}$, then we call $\{Q(t)\}$ a diffusion approximation for $\{C(t)\}$.

The diffusion approximation to the classical risk process is given by Iglehart (1969). To the best of our knowledge, there are a few of papers dealing with diffusion approximations to two-dimensional risk processes in risk theory. In the following, we study the diffusion approximation for the two-dimensional reserve risk process

$$\left(c_1^\pi t - \sum_{i=1}^{N_1(t)+N(t)} f_1(X_i),\ c_2^\pi t - \sum_{i=1}^{N_2(t)+N(t)} f_2(Y_i)\right)$$

and thus obtain the diffusion approximation for the reserve process $\{R^\pi(t)\}$ given in (2.2).

We first construct a sequence of risk reserve processes $\{R_n^\pi(t),\ n = 1, 2, \ldots\}$ using the scale technique, and then give a diffusion approximation of $\{R^\pi(t)\}$. The risk reserve processes $\{R_n^\pi(t)\}$ is constructed as follows.