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Stochastic Pareto-optimal reinsurance policies

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HIGHLIGHTS

- We model reinsurance as a continuous-time stochastic cooperation game.
- We study Pareto-optimal solutions and derive the corresponding HJB equation.
- The optimal policies are classified into two classes of functions.

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ABSTRACT

We model reinsurance as a stochastic cooperation game in a continuous-time framework. Employing stochastic control theory and dynamic programming techniques, we study Pareto-optimal solutions to the game and derive the corresponding Hamilton–Jacobi–Bellman (HJB) equation. After analyzing the HJB equation, we show that the Pareto-optimal policies may be classified into either unlimited excess of loss functions or proportional functions based on different premium share principles. To illustrate our results, we solve several examples for explicit solutions.

1. Introduction

Insurance companies employ reinsurance policies in order to increase underwriting capacity, stabilize profits, or provide protection against a catastrophic loss, etc. A primary insurer that initially writes the insurance business is called a *ceding company* or *cedent*; an insurer that accepts part or all of the insurance from the ceding company is called a *reinsurer*. A reinsurance policy consists of one *risk share function* which determines how the ceding company and the reinsurer share risk, and one *premium share function* which indicates how premium is diverted between the two parties. In this paper, reinsurance policies are determined dynamically in continuous-time. At each moment, a reinsurance policy is selected by the ceding company and the reinsurer. Once the policy is determined, the risk share function is determined. As usual settings in the literature of insurance/reinsurance, the premium share function is assumed to be pre-selected by the two parties.

We work with controls of reinsurance policies under a continuous-time diffusion model which approximates its discrete-time counter party. Continuous-time diffusion models have been used extensively in studies of optimal insurance/reinsurance policies in the literature. In this strand of continuous-time models, Taksar and Markussen (2003) studied optimal reinsurance policies in terms minimizing ruin probabilities. Luo et al. (2008) and Promislow and Young (2005) considered the optimal reinsurance and investment controls to minimize ruin. Zeng (2010a) considered the optimal policies when a rescuing procedure is expected. Luo and Taksar (2011) investigated optimal control of minimizing absolute ruin.

This paper not only follows the strand of research on diffusion reinsurance models but also incorporates ideas of game theory. We note some recent works on related game theory: Golubin (2006) solved one Pareto-optimal insurance game in a static model; Suijs et al. (1998) formulated a cooperation game between insurance and reinsurance with stochastic payoffs; Zeng (2010b) found a Nash equilibrium solution of a reinsurance game between two competing insurers. Such research on risk exchange in insurance and reinsurance can be traced back to Borch (1960), Arrow (1971), and Gerber (1978).

In this paper, we consider a stochastic cooperation game between an insurer and a reinsurer whose surplus processes (on a

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specific segment of business) are described by two diffusion equations. The goal is to seek a Pareto-optimal reinsurance policy that maximizes a weighted sum of two utilities. The utilities are calculated by the two surplus processes accordingly. We note that the value function in the optimization problem involves both surplus processes and the optimization benefits both the insurer and reinsurer. We also note that comparing to zero-sum games, sum of the two surplus processes in this game is an arithmetic Brownian motion. The two players need to split it by setting an optimal reinsurance treaty that maximizes the joint utility. To the best of our knowledge, this paper is the first to connect stochastic control theory and Pareto-optimal cooperation reinsurance games.

The rest of the paper is organized as follows. In Section 2, we introduce the stochastic insurer–reinsurer model and describe the Pareto-optimal problem. The corresponding Hamilton–Jacobi–Bellman equation is derived there as well. In Sections 3 and 4, we study optimal reinsurance policies when the premium share function is based on expected value principle and variance principle respectively. The last section summarizes the paper.

2. The mathematical model

Consider an insurance company whose surplus follows the classical Cramer–Lundberg model

$$X(t) = x + pt - \sum_{i=1}^{N(t)} Z_i,$$

where x is the initial surplus, p is the premium rate, $N(t)$ is the counting Poisson process with the intensity λ, and Z_i is the size of ith claim. Denote by τ_i the time of the ith claim. Z_i, $i = 1, 2, \ldots$ are i.i.d. variables with finite mean μ_0 and variance σ_0. In the classical setting of this model, the premium rate is given by $p = (1 + \eta)\lambda\mu_0$ where η is called *safety loading* representing the additional premium received by the insurer due to the uncertainty.

Now we define risk share functions and premium share functions (the two components of a reinsurance policy) as follows. A *risk share function* $g(z)$ is an increasing function $g : [0, \infty) \to [0, \infty)$ with the property $g(z) \in [0, z]$. For a risk share function $g(\cdot)$, $g(Z_i)$ is the part of each random claim retained by the ceding company while the rest $Z_i - g(Z_i)$ is ceded to the reinsurer. A *premium share function* $\pi(\cdot)$ is a real value function which determines how much premium is diverted to the reinsurer. In particular, as much as $\pi(Z_i - g(Z_i))$ of the premium is diverted to the reinsurer when it picks up the rest of each random claim. More precisely, $\pi(Z_i - g(Z_i))$ represents the rate at which the cedent pays the reinsurer premium.

In this paper, we assume the risk share function can be determined dynamically over the time horizon. We use convention $g(\cdot; t) := g(\cdot)$ to indicate that the risk share policy is determined at each moment in the sequel. A reinsurance policy at time t is denoted by a pair $(g(\cdot; t), \pi(\cdot))$ and under the policy the dynamics of the surplus processes of the ceding company and the reinsurer are given by the following equations:

$$X(t) = x + \int_0^t [p - \lambda\pi(Z - g(Z; s))]ds - \sum_{i=1}^{N(t)} g(Z_i; \tau_i),$$

$$Y(t) = y + \int_0^t [\lambda\pi(Z - g(Z; s))]ds - \sum_{i=1}^{N(t)} (Z - g(Z_i; \tau_i)),$$

where we define $Z := Z_1$.

Without loss of generality, we assume the intensity $\lambda = 1$ in the sequel. The above processes can be approximated by the diffusion processes (see e.g. Klugman et al. (2004)) as follows:

$$dX(t) = \mu_1(t)dt + \sigma_1(t)dW(t), \tag{1}$$

$$dY(t) = \mu_2(t)dt + \sigma_2(t)dW(t), \tag{2}$$

where

$$\mu_1(t) = p - \pi(Z - g(Z; t)) - E[g(Z; t)], \qquad \sigma_1(t) = \sqrt{E[g(Z; t)^2]},$$

$$\mu_2(t) = \pi(Z - g(Z; t)) - E[Z - g(Z; t)], \qquad \sigma_2(t) = \sqrt{E[(Z - g(Z; t))^2]}.$$

We assume $\mu_i(\cdot)$ and $\sigma_i(\cdot)$, $i = 1, 2$ are Lipschitz continuous functions of t. We denote the corresponding probability space by $(\Omega, \mathcal{F}, P)$ endowed with the filtration $\mathcal{F}_t$ and the standard independent Brownian motion $W(t)$ adapted to $\mathcal{F}_t$. A risk share policy g is called *admissible* if the corresponding stochastic differential equations (1), (2) both have strong solutions and $g(z; t)$ is adapted to $\mathcal{F}_t$ for any fixed z. Let $\mathcal{G}$ be the set of all admissible risk share policies.

In this paper we assume the premium share function $\pi(\cdot)$ is exogenous and pre-determined by the ceding company and the reinsurer. Their objective is to find an optimal risk share policy g in terms of maximizing their interests:

$$V_1(t, x, y; g) = E[u_1(X_{T\wedge\tau}, \tau)|X_t = x, Y_t = y, t \le \tau], \tag{3}$$

$$V_2(t, x, y; g) = E[u_2(Y_{T\wedge\tau}, \tau)|X_t = x, Y_t = y, t \le \tau], \tag{4}$$

where T is a fixed time horizon, τ is a stopping time with respect to the filtration $\mathcal{F}_t$, and u_1, u_2 are utility functions of two companies respectively.

An admissible risk share policy $\hat{g} = \{\hat{g}(z; t)\}_{t=0}^T$ is called *Pareto-optimal* if for every $t \in [0, T]$, there is no $g \in \mathcal{G}$ such that

$$V_1(t, x, y; g) \ge V_1(t, x, y; \hat{g}), \qquad V_2(t, x, y; g) \ge V_2(t, x, y; \hat{g}),$$

and at least one of the above inequalities is strict.

For any $\delta \in (0, 1)$, let

$$J^g(t, x, y) := \delta V_1(t, x, y; g) + (1 - \delta)V_2(t, x, y; g). \tag{5}$$

Now define value function

$$V(t, x, y) := \sup_{g\in\mathcal{G}} J^g(t, x, y). \tag{6}$$

It is known (see e.g. Gerber, 1978) that $\hat{g}$ is Pareto-optimal if it solves the following problem of maximizing a sum of weighted utilities:

$$\hat{g} = \arg\max_{g\in\mathcal{G}} \delta V_1(t, x, y; g) + (1 - \delta)V_2(t, x, y; g) \tag{7}$$

for any $\delta \in (0, 1)$. The condition (7) is obviously sufficient for a Pareto-optimal policy. As mentioned by Gerber (1978), it is actually a necessary condition as well if the set $\mathcal{S} := \{(V_1(t, x, y; g), V_2(t, x, y; g)) : g \in \mathcal{G}\}$ is closed and convex in $\mathcal{R}^2$. However, it is hard to verify this assumption in general. And if the assumption fails, the optimal solutions through weighting the utilities may not necessarily include all the Pareto-optimal solutions.

In the objective function (5), δ takes a role of balancing the interests between two parties. One may wonder how the value of δ could be determined. As argued by Golubin (2006), one way to determine δ is given by "experts" exogenously, according to empirical studies. We also observe that when letting $t = T$ in (5), we have

$$\begin{aligned} J^g(T, x, y) &= \delta V_1(T, x, y; g) + (1 - \delta)V_2(T, x, y; g) \\ &= \delta u_1(x, T) + (1 - \delta)u_2(x, T). \end{aligned}$$

Hence δ may be determined by the two parties through relatively weighting their terminal utilities. Other ways may be based on theories of barging games or cooperative games without transferable utilities. For further discussions on the determination of δ, we refer to Golubin (2006) and the references therein.

In particular, in our model δ is considered as a parameter to weight the terminal values of the two parties. Hereinafter, we focus on solving the optimization problem (7) through dynamic programming techniques. And we will characterize the optimal risk share policy for each fixed $\delta \in (0, 1)$ in the following sections.