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Generalized Makeham's formula and economic profitability

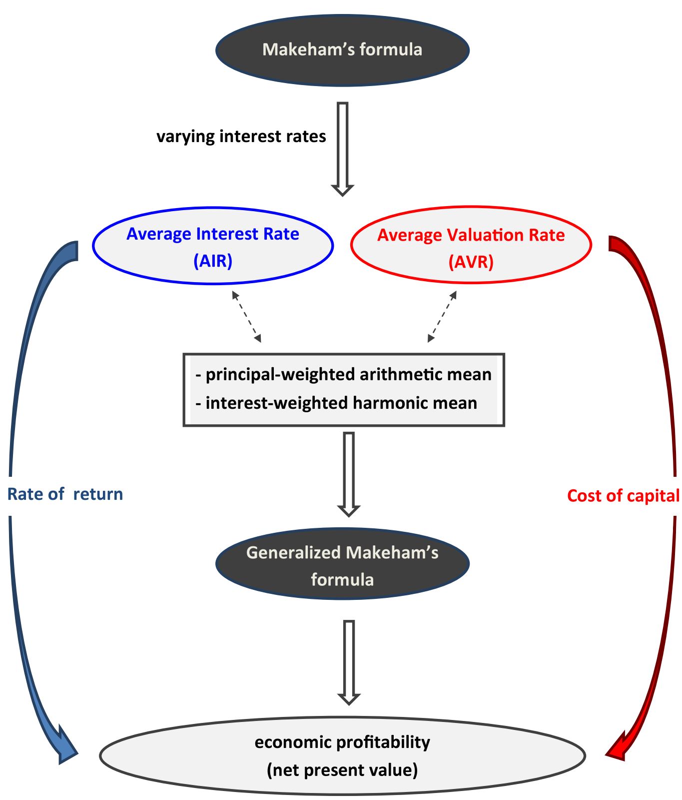
Carlo Alberto Magni *

Department of Economics, University of Modena and Reggio Emilia, CEFIN – Center for Banking and Finance, viale Berengario 51, 41121 Modena, Italy

HIGHLIGHTS

- We present a generalized Makeham's formula allowing for varying interest and valuation rates.
- We derive a unique average interest rate (AIR) and a unique valuation rate (AVR).
- Economic profitability of an asset is captured by comparison of AIR and AVR.
- The AIR and AVR are principal-weighted arithmetic and interest-weighted harmonic means of period rates.
- For portfolios of assets, harmonic and arithmetic means commute, as well as assets and periods.

GRAPHICAL ABSTRACT



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ABSTRACT

This paper generalizes Makeham's formula, allowing for varying interest rates and for a non-flat structure of valuation rates. An *average interest rate* (AIR) is introduced, as well as an *average valuation rate* (AVR), both of which exist and are unique for any asset. They can be computed either as principal-weighted arithmetic means or as interest-weighted harmonic means of period rates. Economic profitability of an asset or a portfolio of assets is captured by the spread between AIR and AVR, which has the same sign as the Net Present Value. This makes (i) AIR a more reliable tool for valuation and decision than the venerable Internal Rate of Return, and (ii) AVR a natural generalization of the cost-of-capital notion.

* Tel.: +39 0592056777.
E-mail address: magni@unimo.it.

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0. Introduction

One of the most interesting relations in the theory of interest has been proposed in nineteenth century by the English actuary and mathematician William Makeham. Named after its begetter, his formula states that the price of a bond can be divided into two components: present value of redemption value plus present value of interest, where the latter is obtained as the ratio of the (possibly modified) coupon rate to the valuation rate times the difference between the redemption value and its present value (Makeham, 1874; Glen, 1893). A more general version of Makeham's formula can be applied to any type of loan, where the coupon rate is replaced by the interest rate of the loan and the difference between redemption value and its present value is replaced by the difference between the borrowed amount and the present value of capital repayments (see Broverman, 2008; Kellison, 2009). While occasionally used in the relatively recent past (Hossack and Taylor, 1975; Ramlau-Hansen, 1988; Astrup Jensen, 1999a,b), Makeham's formula is nowadays essentially neglected in finance and actuarial science, although it directly provides important connections among an asset's value, overall interest and economic profitability expressed as the ratio of two relative measures of worth (interest rate vs. valuation rate). Admittedly, the formula only copes with traditional assets bearing constant interest rate and supplies the above mentioned connections only when the valuation rate is constant. Also, it only copes with financial assets, not with real investments. These features makes it only moderately useful for valuation and decision-making. This paper just aims at generalizing the formula, in such a way that the above mentioned connections are made valid for any kind of assets in any circumstance. In particular, we (i) allow for assets with varying interest rates and consider the more realistic situation where valuation rates vary across time (i.e., the term structure of interest rates is not flat), (ii) extend the application of the formula to any kind of economic activity, including real assets and portfolios of (financial or real) assets, and (iii) provide a valuation/decision tool which is consistent with the net present value (NPV). We find that a suitable weighted mean of the interest rates and a suitable weighted mean of the valuation rates can be used to decompose an asset's value into interest and capital; we call the means 'Average Interest Rate' (AIR) and 'Average Valuation Rate' (AVR), respectively. The term "average" is in a twofold sense: both the AIR and the AVR are principal-weighted arithmetic means of period rates and, at the same time, interest-weighted harmonic means of period rates. While the internal-rate-of-return (IRR) notion suffers from problems of existence and uniqueness and does not guarantee value additivity, the AIR (as well as the AVR) exists and is unique, and the comparison of AIR and AVR correctly captures an asset's economic profitability, while at the same time complying with value additivity.

The paper is structured as follows. Section 1 introduces Makeham's formula and supplies the main definitions. Section 2 generalizes the formula by allowing for varying interest rates: the *average interest rate* (AIR) is introduced, which is shown to exist and be unique. Economic profitability is captured by the yield spread, which is the difference between the AIR and the valuation rate. Section 3 further generalizes the formula by allowing for varying valuation rates: an *average valuation rate* (AVR) is shown to be the correct benchmark for assessment of economic profitability. Section 4 provides a third generalization of Makeham's formula: portfolios of assets are considered, and the portfolio's AIR and AVR are variously expressed as arithmetic or harmonic means of interest rates with the proper weights; it is also shown that the two means enjoy a twofold commutative property. Some concluding remarks end the paper.

1. Makeham's formula

Let $t = 0$ be the current date and let $T_0 = \{0, 1, 2, \dots, n\}$ and $T_1 = \{1, 2, \dots, n\}$.[1] Consider a sequence of cash flows $\{f_t\}_{t\in T_0}$ describing any financial transaction involving two parties which exchange a sequence of monetary amounts by pre-determining an (assumed constant) interest rate i. Following are the well-known relations of an amortization schedule, for $t \in T_1$:

$$f_t = K_t + I_t \tag{1a}$$

$$P_t = P_{t-1} - K_t \qquad f_0 := -P_0 \qquad P_n := 0 \tag{1b}$$

$$I_t = i \cdot P_{t-1}. \tag{1c}$$

P_t is the principal outstanding, also known as capital (outstanding) or outstanding balance, f_t is the payment/disbursement, K_t is the capital payment (principal repayment), I_t is the interest payment, i is the interest rate. All variables are real numbers, with $i > 0$. Let V be the (present) value of cash-flow stream $\{f_t\}_{t\in T_1}$, computed at a valuation rate $r > 0 : V = V(r) = \sum_{t\in T_1} f_t(1+r)^{-t}$. The valuation rate r is the investor's minimum desired rate of return. Assuming that the cash flows are (certain or) expressed as certainty equivalents, r is the *risk-free* rate. Certainty equivalents are the theoretically correct way of dealing with risky cash flows and V represents the asset's arbitrage-free value in a complete market; alternatively, it is possible to discount the asset's expected cash flows at a discount rate that reflects the asset's risk. The latter is often measured by the so-called beta derived from the well-known Capital Asset Pricing Model, so the valuation rate is the return rate of equal-risk (i.e., equal beta) alternatives traded in the market, which means that V is the mean–variance value of the asset.[2]

The *internal rate of return* (IRR) is a discount rate x such that the present value of payments equals the present value of disbursements. Note that the interest rate i is the IRR of $\{f_t\}_{t\in T_0}$, since (1) implies $\sum_{t\in T_0} f_t(1+i)^{-t} = 0$. Likewise, the valuation rate r is the IRR of the asset $(-V, f_1, \dots, f_n)$.

By (1a), one may divide the value of the asset into an interest portion $\mathcal{I}$ and a capital portion $\mathcal{K}$:

$$V = \mathcal{I} + \mathcal{K} \tag{2}$$

where $\mathcal{I} = \mathcal{I}(i, r) := \sum_{t\in T_1} i \cdot P_{t-1}(1+r)^{-t} = \sum_{t\in T_1} I_t v^t$ is the (present) value of the interest portion and $\mathcal{K} = \mathcal{K}(i, r) := \sum_{t\in T_1} (f_t - i \cdot P_{t-1}) \cdot (1+r)^{-t} = \sum_{t\in T_1} K_t v^t$ is the (present) value of the capital portion, and $v := (1+r)^{-1}$ is the discount factor. Makeham's formula relates $\mathcal{I}$ and $\mathcal{K}$ in the following way:

$$\mathcal{I} = \mathcal{I}(i, r) = \frac{i}{r}(P_0 - \mathcal{K}) \tag{3a}$$

so that

$$V = V(i, r) = \frac{i}{r}(P_0 - \mathcal{K}) + \mathcal{K}. \tag{3b}$$

Economic profitability of an asset depends on a comparison between value and borrowed amount or, equivalently, on the sign of the Net Present Value (NPV). An asset is economically profitable (i.e., wealth is increased) if

$$V > P_0 \iff \text{NPV}(r) > 0. \tag{4}$$

The NPV measures the investor's wealth increase, with respect to the preference rate, which is also known as *cost of capital* in corporate finance. It is evident that $i = r$ implies $V = P_0$, which means $\text{NPV} = 0$.

[1] Throughout the paper, we use the set notations $\sum_{t\in T_0}$, $\sum_{t\in T_1}$, etc., for in-text summations.

[2] For relations between mean–variance pricing and arbitrage-free pricing see Dybvig and Ingersoll (1982) and Magni (2009).