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Modeling future lifetime as a fuzzy random variable

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HIGHLIGHTS

- This article models future lifetime as a fuzzy random variable (FRV).
- We begin by discussing the motivation for the study.
- Next we provide a brief review of future lifetime as a random variable (RV).
- This is followed by a discussion of future life time as a fuzzy variable (FV).
- Finally, we merge the RV with the FV to model future lifetime as a FRV.

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ABSTRACT

A recent article by de Andrés-Sánchez and Puchades (2012) modeled life annuities as fuzzy random variables (FRVs). Their article was informative. However, it had the limitation that the FRV used to model the life annuity was not a granulated FRV. This followed because the authors assumed that the uncertainty insofar as mortality is entirely due to randomness and that the uncertainty with respect to interest rates is entirely due to fuzziness. The concern is that such a dichotomy may be problematic since, in actuality, the uncertainty of both the mortality parameter and the interest rate parameter can have both random and fuzzy features. The purpose of this article is to address the mortality portion of this dichotomy and, to this end, we model future lifetime as a FRV.

1. Introduction

A recent article by de Andrés-Sánchez and Puchades (2012) modeled life annuities as fuzzy random variables (FRVs). Their article was informative: it discussed the roles of mortality and interest in the development of a life annuity, it explicitly presented the mechanics of putting randomness and fuzziness together to form a FRV, and it showed how to formulate an aggregate model of a portfolio of annuities. However, a limitation of their article was that the FRV used to model the life annuity was not a granulated FRV, where a granulated FRV is a FRV whose potential FRV parameters, if any, are explicitly modeled as such. On the contrary, it was assumed that the uncertainty insofar as the mortality parameter is entirely due to randomness and that the uncertainty with respect to interest rate parameter is entirely due to fuzziness. Then the stochastic mortality rates were merged with the fuzzy discount rates to formulate the life annuity as an ungranulated FRV.

As the authors noted, while a number of studies have focused on the stochastic nature of mortality, other studies, including some mentioned by the authors, such as Lemaire (1990), have modeled the fuzzy nature of mortality. Similarly, while a number of studies have focused on the fuzzy nature of interest rates, other studies, such as Panjer and Bellhouse (1981), have focused on its stochastic nature. The point is that the uncertainty of both the mortality parameter and the interest rate parameter has both random and fuzzy features, so they are both potential FRVs. Hence, the dichotomy suggested by the authors, and the resulting ungranulated FRV, may be problematic, particularly if their model is to be implemented in practice.

The purpose of this article is to address the mortality portion of the foregoing dichotomy.[1] To this end, we model future lifetime as a FRV. Moreover, in keeping with the demographic context of de Andrés-Sánchez and Puchades (2012), we focus on the future lifetime of retirees.[2]

We start by considering the random variable future lifetime of a life aged x, $T(x)$. The analytical nature of $T(x)$ is discussed

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[1] This is not to suggest that the interest portion should not be revisited. That analysis, however, is deferred to a future study.

[2] As a class, retirees form an age group for whom the life annuities discussed by de Andrés-Sánchez and Puchades (2012) are relevant.

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in actuarial texts like Bowers et al. (1997, 52), Dickson et al. (2009, 17),[3] and Gerber (1997, 15) while its application in a post-retirement context is explored in articles such as Babbel and Merrill (2007), Brown (2004), Horneff et al. (2008), Kapur and Orszag (2002), Milevsky (2004) and Young (2004).

The foregoing citations often concentrate on the relationship between attained age, x, and $T(x)$, and the implications of that relationship. However, while age is an important factor in the determination of $T(x)$, there are other relevant factors.[4] Moreover, some of the other dominant factors, like the state of health and the character of a life aged x, are often encapsulated in a perceived fuzzy metric, like "less than average future lifetime". Thus, $T(x)$ might more appropriately be written as $\tilde{T} = \tilde{T}(x|\tilde{f})$, where the tilde denotes a fuzzy parameter and $\tilde{f}$ represents a fuzzy metric other than age, in which case $\tilde{T}$ can be conceptualized as a FRV (Shapiro, 2009). The purpose of this article is to explore this conceptualization of future lifetime as a FRV.

The article proceeds as follows. We begin with a simple statement of the problem. This is followed by a brief overview of future lifetime as a random variable, where, for simplicity, we assume the Gompertz form of the force of mortality. The parameters are chosen so that the expected future lifetime at age 65 is 15 years. Next comes a short section on linguistic variables, membership functions (MFs), and methods for assigning MFs to fuzzy variables. Then, MFs for short, average and long future lifetime are discussed. This is followed by a discussion of future lifetime as a FRV. The article ends with a comment on potential applications involving future lifetime as a FRV and areas for potential refinements of the model.

2. A statement of the problem

To put the problem in a context, consider the task of giving post-retirement financial planning advice to a new retiree. Assume that after a discussion with the retiree, it is concluded that she is a standard life,[5] and, at this juncture, the talking point is her future lifetime. To facilitate this discussion, we choose from the linguistic scale $\mathbb{L}$, which is composed of the terms "short future lifetime", "medium future lifetime", and "long future lifetime".[6] Each of these labels can be viewed as a fuzzy subset of the future lifetime scale. This information can be described by a FRV $\tilde{T} : \Omega \to \mathbb{L}$, where Ω is the set of all possible new retirees, each $\omega \in \Omega$ represents a new retiree with a particular set of features, and $\tilde{T}(\omega)$ represents the label assigned to his or her future lifetime (short, medium or long).

This scenario is an example of the Puri and Ralescu (1986) view of FRVs, where the fuzzy variable, a fuzzy subset of the future lifetime scale, is associated with the randomly chosen new retiree.

3. Future lifetime as a random variable

The simplest (classical) approach to modeling future lifetime is to model it in terms of an n-year period certain, where the period certain might be the expected future lifetime, for example.[7] The model is given a stochastic dimension by reformulating future lifetime as a random variable (RV), $T(x) \equiv X - x$, where X and x are the random age at death and current age, respectively, of an individual. This can be conceptualized as shown in Fig. 1.

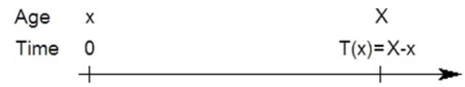

Fig. 1. The random variable future lifetime, $T(x)$.

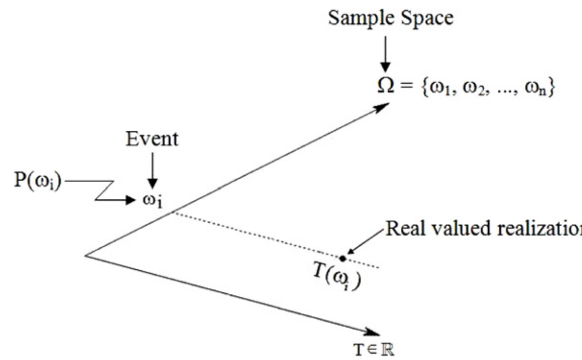

Fig. 2. RV trajectory.

More formally, let $(\Omega, \mathcal{A}, P)$ be a probability space, where Ω is sample space, the set of all possible events or outcomes, $\{\omega_i\}, i = 1, \ldots, n$, $\mathcal{A}$ is an event space (also called a σ-algebra or σ-field), the set of all possible potentially interesting events, and P is a probability measure[8] over $\mathcal{A}$, that is, ω_i is sampled from Ω according to probability measure P. This characterization of the dynamics of the probability space is depicted in Fig. 2, labeled RV trajectory, which also shows the realization of the future lifetime of a life aged x with the ith set of features, $T(\omega_i) \equiv T(x|\omega_i)$.

As is well known, the pdf of $T(x)$ generally is expressed in the form

$$f_{T(x)}(t) = {}_t p_x \, \mu_{x+t}, \quad t \geq 0, \tag{1}$$

where ${}_t p_x$ represents the conditional probability that a life aged x will survive for t more years, having survived to age x, and μ_{x+t} represents the instantaneous force of mortality at age $x + t$.[9] This latter is analogous to the hazard rate in reliability theory. Since $\mu_{x+t} = -\frac{d}{dt} \ln {}_t p_x$,[10]

$${}_t p_x = e^{-\int_0^t \mu_{x+s} ds}. \tag{2}$$

The cumulative distribution of the future lifetime of a life aged x takes the form

$$F_{T(x)}(t) = \int_0^t f_{T(x)}(t) ds = \int_0^t {}_s p_x \, \mu_{x+s} ds \tag{3}$$

and the expected future lifetime can be expressed as

$$\mathring{e}_x = E\{T(x)\} = \int_0^\infty t \, {}_t p_x \, \mu_{x+t} dt, \tag{4}$$

where the upper limit of ∞ on the integrals indicates that the integration is over all positive probability density.

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3 Dickson et al. (2009, 17) in contrast to other actuarial texts, use T_x to denote the random variable future lifetime.

4 From an insurer perspective, factors that would lead to a life expectancy perception with respect to an individual include such things as blood pressure, smoking status, cholesterol ratios, build, driving record, and family history of cancer. See Justman (2007) for a discussion of these factors. From the perspective of the retiree, life expectancy perception is based on family history, current health status, healthy habits (staying active, exercising, eating right, and not smoking), positive attitude (no stress, no worries, desire to see grandchildren), average life expectancy, and good health care (Society of Actuaries, 2012).

5 A person who is a standard life would meet an insurer's underwriting criteria for a standard policy, that is, a policy issued with standard provisions and at standard rates.

6 This linguistic scale terminology was adapted from Couso and Dubois (2009, 1072).

7 King (1902, 112) discussed the error inherent in this approach.

8 $P(\Omega) = 1$, $P(A) \geq 0$ for any $\mathcal{A} \in \mathcal{A}$, and P is countably additive.

9 See Bowers et al. (1997, 52), Dickson et al. (2009, 17), and Gerber (1997, 15). Following these actuarial textbooks and articles like de Andrés-Sánchez and Puchades (2012), we take the force of mortality as given.

10 Bowers et al. (1997, 56).