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Optimal proportional reinsurance and investment with regime-switching for mean–variance insurers

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HIGHLIGHTS

- An optimal investment–reinsurance problem under mean–variance criterion is considered.
- The market is regime-switching.
- We require no short-selling restriction in the investment policy.
- All the parameters in our model are time-varying and regime-switching.
- The Maximum Principle in the optimal control is applied.

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ABSTRACT

Following the framework of Promislow and Young (2005), this paper considers an optimal investment–reinsurance problem of an insurer facing a claim process modeled by a Brownian motion with drift under the Markowitz mean–variance criterion. The market modes are divided into a finite number of regimes. All the key parameters change according to the value of different market modes. The insurer chooses to purchase proportional reinsurance to reduce the underlying risk. In addition to reinsurance, we suppose that the insurer is allowed to invest its surplus in a financial market consisting of a risk-free asset (bond or bank account) and a risky asset whose price process is modeled by a geometric Brownian motion. We investigate the feasibility of the problem, obtain an analytic expression for the optimal strategy, delineate the efficient frontier and demonstrate our results with numerical examples.

1. Introduction

Reinsurance is an insurance protection that is purchased by an insurance company (the insurer which is also called a "cedant" or "cedent") from a reinsurer as a means of risk management. It is a transfer of risk from the direct insurer to the secondary insurance carrier. The best known examples of reinsurance contracts are stop loss, proportional and excess of loss arrangements. Reinsurance is one of the main business activities to control the underlying risks of an insurance company, however, the insurer's investment in some risky assets is another important source of risk control. In this paper, we incorporate both reinsurance and investment in an optimal portfolio selection problem faced by an insurance company, namely, the optimal investment–reinsurance problem.

Problems with these features can be studied under a variety of objectives, to name a few: minimizing the probability of ruin (see Promislow and Young, 2005, Chen et al., 2010); maximizing the utility of the terminal wealth (see Cao and Wan, 2009, Liu and Ma, 2009, Liang et al., 2011); maximizing the minimal expected exponential utility of terminal wealth over a family of real-world probability measures (see Zhang and Siu, 2009); minimizing the maximal expected discounted penalty of ruin (see Zhang and Siu, 2009).

In recent years, the mean–variance criterion pioneered by Markowitz (1952) has been studied by many scholars in connection with optimal investment–reinsurance problems. Bai and Zhang (2008) solved optimal reinsurance/new-business and investment (no-shorting) strategies for the mean–variance problem in two risk models: a classical risk model and a diffusion model. The insurer can invest in both risk-free and risky assets in addition to purchasing reinsurance/new-business. The efficient frontiers and efficient strategies are derived explicitly by the verification theorem for the viscosity solutions of the corresponding Hamilton–Jacobi–Bellman (HJB) equations. Bi et al. (2011) extended their classical model to the case with investment in multiple risky assets. Zeng and Li (2011) considered optimal time-consistent investment and reinsurance policies for mean–variance insurers.

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Meanwhile regime switching models have become popular in economics, finance and actuarial science. This type of model is motivated by the intention of reflecting various states of the financial market. For example, the market status can take either one of two regimes: bullish or bearish, in which the price movements of the stocks could be quite different. Generally, in a regime-switching model, the market mode can take values in one of a finite number of regimes. The key parameters, such as the bank interest rate, or stocks appreciation and volatility rates, will change according to the value of different market modes. Since the market state may change from one regime to another, both the nature of the regime and the change point should be estimated. In literatures, if the market state process is modeled by a continuous time Markov chain with finite states, regime switching models are also referred to as Markov switching or Markov-modulated models.

With time-varying parameters, regime switching models are obviously more realistic than models with constant parameters, since they better reflect the random nature of the underlying market environment. As discussed in Neftci (1984), an appealing property of these models is to account for the accumulating evidence that business cycles are asymmetric. Most of the studies indicate that regime-switching models perform well in some sense, for example, Hardy (2001) used monthly data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices to fit a regime switching lognormal model. In her work, the fit of regime switching models to the data was compared with other econometric models, and she found that regime-switching models provided a significant improvement over all other models in the sense of maximizing the likelihood function. In the special case of a lognormal setting, the Excel based software "Regime Switching Equity Model Workbook" developed by Hardy and her group (which is available on the Society of Actuaries website: www.soa.org/professional-interests/investment/invest-regime-switching-equity-model-workbook-version-1-updated.aspx) can be applied directly, which greatly simplifies the implementation procedure of regime-switching models.

Regime-switching models are not new in statistics and economics, dating back to Henderson and Quandt (1958), where regime regression models were first investigated. Kim and Nelson (1999) gave a brief review of Markov switching models and presented a comprehensive exposition of statistical methods for these models as well as many empirical studies. One influential work on the application of regime switching models is Hamilton (1989), where dynamic models with Markov switching between regimes were introduced as a tool for dealing with endogenous structural breaks. And after that, several contributions in the economic literature have replied on regime switching structure including models for business cycle asymmetry, see Hamilton (1989) and Lam (1990); the effects of oil prices on US GDP growth, see Raymond and Rich (1997); and labor market recruitment, see Storer and Van Audenrode (1995).

However, it is not until the recent years that applications of regime switching models have been established in quantitative finance and insurance. Early works are done on option pricing, see Di Masi et al. (1994), Buffington and Elliott (2002) and Boyle and Draviam (2007). After that, regime switching models were applied to many other aspects, such as equity-linked life insurance pricing, see Hardy (2003); bond pricing, see Elliott and Siu (2009a); portfolio selection, see Zhou and Yin (2003), Guidolin and Timmermann (2007), Chen et al. (2008) and Elliott and Siu (2009b); optimal dividend, Li and Lu (2006, 2007), etc.

In this paper, we consider an optimal investment–reinsurance problem under the mean–variance criterion in a regime-switching market. The structure of this paper is similar to Zhou and Yin (2003). Section 2 provides the formulation of the problem. Section 3 discusses the feasibility of the problem. The optimal strategy is established in Section 4. The efficient frontier is constructed in Section 5. Section 6 gives some numerical results and we conclude in Section 7.

Comparing to Zhou and Yin (2003), we require a no short-selling restriction in the investment policy which makes the problem harder in the sense that the resulting optimal investment strategy by the Maximum Principle may not satisfy the restriction. Besides, we add reinsurance into the problem, therefore we need to model a reinsurance policy, see the definition for admissibility in Section 2. Hence their results no longer hold in the new setting. We illustrate the differences in the relevant sections.

2. Problem formulation

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a filtered complete probability space on which we define the standard Brownian motion $W(t) = (W_0(t), W_1(t))$ and a continuous-time stationary Markov chain $\alpha(t)$ taking value in a finite state space $\mathcal{M} = \{1, 2, \ldots, d\}$ such that $W(t)$ and $\alpha(t)$ are independent of each other. The Markov chain has a generator $Q = (q_{ij})_{d\times d}$ and stationary transition probabilities:

$$p_{ij}(t) = P(\alpha(t) = j|\alpha(0) = i), \quad t \geq 0,\ i, j = 1, 2, \ldots, d.$$

Following the framework of Promislow and Young (2005), we model the claim process $C(t)$ according to a Brownian motion with drift as

$$dC(t) = a(t, \alpha(t))dt - b(t, \alpha(t))dW_0(t) \tag{2.1}$$

in which $a(t, i)$ and $b(t, i)$ are positive processes according to market mode $i \in \mathcal{M}$. We assume that the premium is paid continuously at rate $c_0(t, i) = (1 + \theta(t, i))a(t, i)$ with safety loading $\theta(t, i) > 0$ according market mode $i \in \mathcal{M}$. Then before introducing reinsurance and the investment, the surplus process $R(t)$ is given by

$$dR(t) = c_0(t, \alpha(t))dt - dC(t). \tag{2.2}$$

To reduce the underlying risk, the insurer chooses to purchase proportional reinsurance. If the total claim is denoted by Y, we assume $q(t)Y$ is reinsured where $q(t)$ is the proportion at time t. The insurer (or cedent) pays reinsurance premiums continuously at rate $c_1(t, i) = (1 + \eta(t, i))a(t, i)q(t)$ with safety loading $\eta(t, i)$ according to market mode $i \in \mathcal{M}$. The proportional reinsurance is called cheap if $\eta = \theta$ while being not cheap if $\theta < \eta$, see Zeng and Li (2011). In this paper we consider non-cheap reinsurance, that is, $\eta(t, i) > \theta(t, i) > 0$ for any market mode $i \in \mathcal{M}$. Then at time t the cedent will pay $100(1 - q(t))\%$ of each claim while the rest of $100q(t)\%$ is paid by the reinsurer. After the purchase of reinsurance, the surplus process $R(t)$ becomes

$$\begin{aligned} dR(t) &= c_0(t, \alpha(t))dt - (1 - q(t))dC(t) - c_1(t, \alpha(t))dt \\ &= (\theta(t, \alpha(t)) - \eta(t, \alpha(t))q(t))a(t, \alpha(t))dt \\ &\quad + b(t, \alpha(t))(1 - q(t))dW_0(t). \end{aligned} \tag{2.3}$$

In addition to purchasing reinsurance, we suppose that the insurer is allowed to invest its surplus in a financial market consisting of a risk-free asset (bond or bank account) and a risky asset. Let $S_0(t)$ denote the price process of the risk-free asset which is modeled by

$$dS_0(t) = r_0(t, \alpha(t))S_0(t)dt, \quad r_0(t, i) > 0,\ i = 1, 2, \ldots, d. \tag{2.4}$$

For the risky asset, we use Markov-modulated geometric Brownian motion to describe its price process. Let $S_1(t)$ denote the price process of the risky asset with dynamics given by

$$\begin{aligned} &dS_1(t) = r_1(t, \alpha(t))S_1(t)dt + \sigma(t, \alpha(t))S_1(t)dW_1(t), \\ &\quad i = 1, 2, \ldots, d, \end{aligned} \tag{2.5}$$

where $r_1(t, i)$ is the appreciation rate and $\sigma(t, i)$ is the volatility or the dispersion rate according to market mode $i \in \mathcal{M}$.