Insurance: Mathematics and Economics 53 (2013) 906–918

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Dividend problems in the dual risk model

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HIGHLIGHTS

- We consider the compound Poisson dual risk model with a dividend barrier strategy.
- We establish a closed connection between the dual and the classical risk models.
- We present a new approach for the calculation of expected discounted dividends.
- We study ruin and dividend probabilities, number of dividends and time to a dividend.
- We also present the distribution for the amount of single dividends.

ARTICLE INFO

Article history:
Received July 2011
Received in revised form
August 2013
Accepted 7 October 2013

Keywords:
Dual risk model
Classical risk model
Ruin probabilities
Dividend probabilities
Discounted dividends
Dividend amounts
Number of dividends

ABSTRACT

We consider the compound Poisson dual risk model, dual to the well known *classical* risk model for insurance applications, where premiums are regarded as costs and claims are viewed as profits. The surplus can be interpreted as a venture capital like the capital of an economic activity involved in research and development. Like most authors, we consider an upper dividend barrier so that we model the gains of the capital and its return to the capital holders.

By establishing a proper and crucial connection between the two models we show and explain clearly the dividends process dynamics for the dual risk model, properties for different random quantities involved as well as their relations. Using our innovative approach we derive some already known results and go further by finding several new ones. We study different ruin and dividend probabilities, such as the calculation of the probability of a dividend, distribution of the number of dividends, expected and amount of dividends as well as the time of getting a dividend.

We obtain integro-differential equations for some of the above results and also Laplace transforms. From there we can get analytical results for cases where solutions and/or inversions are possible, in other cases we may only get numerical ones. We present examples under the two cases.

1. Introduction

We consider in this manuscript the dual risk model, as described, for instance, by Avanzi et al. (2007). The surplus or equity of a company at time t is given by the equation,

$$U(t) = u - ct + S(t), \quad t \geq 0, \tag{1.1}$$

where u is the initial surplus, c is the constant rate of expenses, $\{S(t), t \geq 0\}$ is a compound Poisson process with parameter λ and density function $p(x)$, $x > 0$, of the positive gains, with mean p_1 (we therefore assume that it exists). Its distribution function is denoted as $P(x)$. The expected increase per unit time, given by $\mu = \mathbb{E}[S(1)] - c = \lambda p_1 - c$, is positive, that is $c < \lambda p_1$. All these quantities have a corresponding meaning in the well known classical continuous time risk model, also known as the Cramér–Lundberg risk model, for insurance applications. For the remainder of our text we will refer to this latter model as simply the standard risk model (shortly SM). For those used to working with it we note that the income condition, $c < \lambda p_1$, is reversed. A few authors have addressed the dual model (simply DM), we can go back to Gerber (1979, pp. 136–138) who called it the *negative claims model*, also see Bühlmann (1970). We can go even further back to authors like Cramér (1955), Takács (1967) and Seal (1969).

Avanzi et al. (2007, Section 1), explains well where applications of the dual model are said to be appropriate. We just retain a simple but illustrative interpretation, the surplus can be considered as the capital of an economic activity like research and development where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity and gets occasional profits according to a Poisson process. This model

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http://dx.doi.org/10.1016/j.insmatheco.2013.10.003

has been recently used by Bayraktar and Egami (2008) to model capital investments. Indeed, recently the model has been targeted with several developments, involving the present value of dividend payments and/or dividend strategies. We underline the cited work by Avanzi et al. (2007) and Avanzi (2009), an excellent review paper. Other works are of importance, some of which we briefly review below.

Important financial applications of the model ruled by (1.1) are the modeling of future dividends of the investments. So, we add an upper barrier, the dividend barrier, noted as $b\,(\geq u \geq 0)$. We refer to the upper graph in Fig. 1 (see also Fig. 1 of Avanzi et al., 2007). On the instant the surplus upcrosses the barrier a dividend is immediately paid and the process restarts from level b. We can also consider the case $b < u$, however an immediate dividend is paid and the process starts from b, see Avanzi et al. (2007). This makes the situation less interesting from our point of view, so we will concentrate our work to the case $u \leq b$.

In this manuscript we are not interested on strategies of dividend payments, we focus on some key quantities, given a barrier level b. We will consider the payments either discounted or not. Several papers have been published recently using this model with an upper dividend barrier, where the calculation of expected amounts of the discounted paid dividends is targeted. Higher moments have also been considered. See Avanzi et al. (2007), Avanzi and Gerber (2008), Cheung and Drekic (2008), Gerber and Smith (2008) and Ng (2009, 2010). Yang and Zhu (2008) compute bounds for the ruin probability. Song et al. (2008) consider Laplace transforms for the calculation of the expected duration of negative surplus. Cheung (2012) also deals with negative surplus excursion related problems.

For those works as well as in ours where the dividend barrier b is the key point, it is important to emphasize two aspects: we are going to consider two barriers, one reflecting and another absorbing, the dividend barrier b and the ruin level "0", respectively. In the case of the upper barrier b the process restarts at level b if this is overtaken by a gain. As mentioned above, this is because an immediate amount of surplus in excess of b is paid in the form of a dividend, it is a *pay-back* capital. It is not the case with the *ruin level* which makes the process die down. Indeed, this happens with probability one (we will come back to this issue later in the text). To achieve a payable dividend the process must not be ruined previously. Furthermore, under the conditions stated the process, *sooner or later*, will reach one of the two barriers, we mean, with probability one the process reaches a barrier.

In this paper we focus on the connection between the SM and the DM, and based on this we work on unknown problems, however having present some known results from a different viewpoint, which in some cases have interesting interpretations. We will underline these points appropriately. We base our research on the insights and ideas known from the classical risk model. This is a key point for our research. We first do a brief survey of the known results from the literature, then we make important connections between the classical and the dual model features. Afterwards, we make our own developments. We consider important that known results can be looked at from our point of view so that our further developments are better taken and understood.

Let us now consider some of the basic definitions and notation for the dual risk model, those which we address throughout this paper. Some specific quantities we will define and denote on the appropriate section only. First, consider the process as driven by Eq. (1.1), free of the dividend barrier. Let

$$\tau_x = \inf\{t > 0 : U(t) = 0 | U(0) = x\},$$

be the time to ruin, this is the usual definition for the model free of the dividend barrier ($\tau_x = \infty$ if $U(t) \geq 0\,\forall t \geq 0$). Let

$$\psi(x, \delta) = \mathbb{E}\left[e^{-\delta\tau_x} I(\tau_x < \infty) | U(0) = x\right],$$

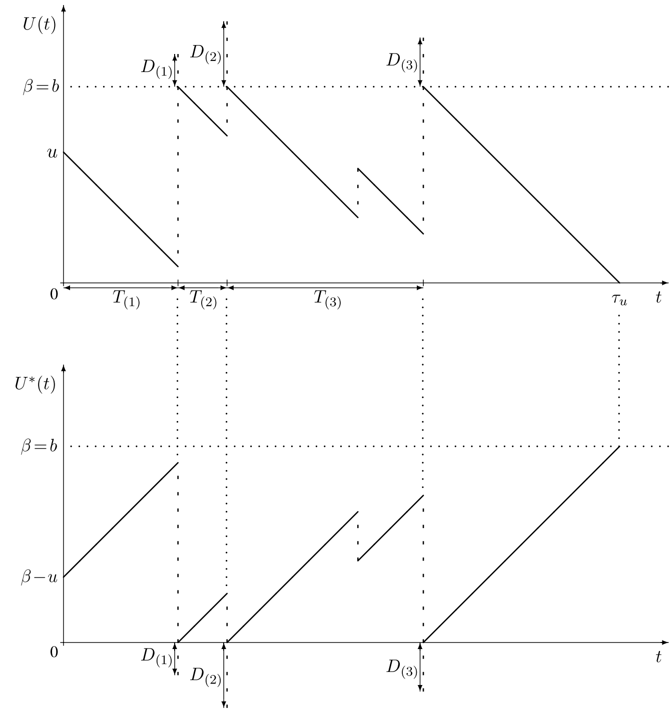

Fig. 1. Classical *vs.* dual model.

where δ is a non negative constant. $\psi(u, \delta)$ is the Laplace transform of time to ruin τ_x. If $\delta = 0$ it reduces to the probability of ultimate ruin of the process free of the dividend barrier, when $\delta > 0$ we can see $\psi(u, \delta)$ *as the present value of a contingent claim of one payable at* τ_x, *evaluated under a given valuation force of interest* δ (see Ng, 2010).

Let us now consider an arbitrary upper level $\beta \geq u \geq 0$ in the model, see the upper graph of Fig. 1, we do not call it yet a dividend barrier. Let

$$T_x = \inf\{t > 0 : U(t) > \beta | U(0) = x\}$$

be the time to reach an upper level $\beta \geq x \geq 0$ for the process which we allow to continue even if it crosses the ruin level "0". Due to the income condition, T_x is a proper random variable since the probability of crossing β is one.

Let us now introduce into the model the barrier $\beta = b$ as a dividend barrier, and the ruin barrier "0", respectively reflecting and absorbing, such that if the process is not ruined it will reach the level b. Here, an immediate dividend is paid by an amount in excess of b, the surplus is restored to level b and the process resumes. We will be mostly working the case $0 < u \leq b$. Dividend will only be due if $T_x < \tau_x$ and ruin will only occur prior to that upcross otherwise. Whenever we refer to conditional random variables, or distributions, we will denote them by adding a "tilde", like $\tilde{T}_x$ for $T_x | T_x < \tau_x$.

Let $\chi(u, b)$ denote the probability of reaching b before ruin occurring, for a process with initial surplus u, and $\xi(u, b) = 1 - \chi(u, b)$ is the probability of ruin before reaching b. We have $\chi(u, b) = \Pr(T_u < \tau_u)$.

Because of the existence of the barrier b ultimate ruin has probability one. The ruin level can be attained before or after the process is reflected on b. Then the probability of ultimate ruin is $\chi(u, b) + \xi(u, b) = 1$.

Let $D_u = \{U(T_u) - b \text{ and } T_u < \tau_u\}$ be the dividend amount and its distribution function be denoted as

$$G(u, b; x) = \Pr(T_u < \tau_u \text{ and } U(T_u) \leq b + x | u, b)$$

with density $g(u, b; x) = \frac{d}{dx}G(u, b; x)$. $G(u, b; x)$ is a defective distribution function, clearly $G(u, b; \infty) = \chi(u, b)$.