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On iterative premium calculation principles under Cumulative Prospect Theory

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HIGHLIGHTS

- The paper reminds the iterativity property and its application.
- We study when the zero-utility principle under Cumulative Prospect Theory is iterative.
- The sufficient and necessary conditions for iterativity are stated.

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ABSTRACT

In the paper we analyze the iterativity condition for zero utility principle adjusted to Cumulative Prospect Theory. We prove, under mild conditions, that the premium principle is iterative if and only if the value function is linear or exponential and probability distortion functions are identities, i.e. the probabilities are not distorted.

1. Cumulative Prospect Theory and zero utility principle

In the rank-dependent utility model it is assumed that probabilities are distorted by some increasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$, called probability distortion function (e.g. Segal, 1989, Denneberg, 1994). Let $\mathcal{G}$ denote the class of all probability distortion functions. For a fixed $g \in \mathcal{G}$ and non-negative random variable X, the Choquet integral is defined by

$$E_g X := \int_0^\infty g\left(P\left(X > t\right)\right) dt.$$

Further we assume that all random variables are defined on some probability space $(\Omega, \mathcal{A}, P)$. If X takes a finite number of values $x_1 < x_2 < \cdots < x_n$ with probabilities $P(X = x_i) = p_i > 0$, then $E_g X = x_1 + \sum_{i=1}^{n-1} g(q_i)(x_{i+1} - x_i)$, where $q_i = \sum_{k=i+1}^{n} p_k$; in particular for $n = 2$ we have $E_g X = x_1 (1 - g(p_2)) + g(p_2) x_2$.

For $g, h \in \mathcal{G}$ and an arbitrary random variable X we define the generalized Choquet integral as

$$E_{gh} X = E_g X_+ - E_h (-X)_+ ,$$

provided that both integrals are finite. Here and subsequently, $X_+ = \max\{0, X\}$. The generalized Choquet integral is introduced by Tversky and Kahneman (1992) for discrete random variables and is used to describe the mathematical foundations of Cumulative Prospect Theory. In numerous experiments Tversky and Kahneman notice that probabilities of losses are distorted in a different way than probabilities of gains. They suggest replacing the utility function with a value function that depends on relative

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payoff. In contrast to expected utility theory, the value function measures losses and gains but not absolute wealth. Under Cumulative Prospect Theory, both value function as well as probability distortion function do not have to be differentiable.

Now, we remind a premium principle which is a modification of the zero-utility principle adjusted to Cumulative Prospect Theory. Let X be a non-negative random variable. Consider an insurance company which has a reference point $w \geq 0$ (e.g. initial wealth) and which wants to sell an insurance policy paying out the monetary equivalent of the random loss X. Further, we call $(X-w)_+$ losses (or catastrophic losses) and $(w-X)_+$ gains (or non-catastrophic losses). In the latter case there is a direct analogy with stop-loss reinsurance. Assume that $u_1, u_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are some strictly increasing value functions, where u_1 measures gains and u_2 measures losses. Let g and h be probability distortion functions of gains and losses, respectively. Kaluszka and Krzeszowiec (2012) introduce the premium $H(X)$ for insuring X as the solution of

$$u_1(w) = E_g u_1\left((w+H(X)-X)_+\right) - E_h u_2\left((X-w-H(X))_+\right). \quad (1)$$

Notice that (1) can be rewritten as

$$u(w) = E_{gh} u\,(w+H(X)-X) \quad (2)$$

with strictly increasing function $u(x) = u_1(x_+) - u_2\left((-x)_+\right)$ for $x \in \mathbb{R}$. Gerber (1979) considers a similar equation for premium $H(X)$ under the assumptions that the value function u is concave and probabilities are not distorted, i.e. $g(p) = h(p) = p$. In a more general model, Heilpern (2003) assumes that $h(x) = 1-g(1-x)$, g is convex and the value function is concave. Van der Hoek and Sherris (2001) analyze a functional with different probability distortion functions for gains and losses. However, they study only the case when the value functions are linear. Goovaerts et al. (2010) consider a risk measure obtained by applying the equivalent utility principle in rank-dependent utility and analyze when such defined risk measure is additive. It turns out that the risk measure introduced by them corresponds to the premium $H(X)$ determined from (1) under $w = 0$. Further, we assume that u is continuous, strictly increasing and $u(0) = 0$. If u is linear or exponential, then after rewriting (2), we can obtain explicit formulas for $H(X)$ (see Kaluszka and Krzeszowiec, 2012).

We define the conditional generalized Choquet integral as

$$E_{gh}(X|Y) = \int_0^\infty g\left(P\left(X_+ > s|Y\right)\right) ds - \int_0^\infty h\left(P\left((-X)_+ > s|Y\right)\right) ds,$$

if both integrals are finite. Then $H(X|Y)$ is introduced as the solution of

$$u(w) = E_{gh}\left[u\left(w+H(X|Y)-X\right)|Y\right].$$

A premium principle $H(X)$ is said to be iterative, if

$$H(X) = H\left(H(X|Y)\right)$$

for all random variables X and Y, provided that both $H(X)$ and $H(H(X|Y))$ exist.

The concept of iterativity dates back at least to Bühlmann (1970), who explains the difference between risk (individual) and collective premium. In order to calculate an individual premium, an insurer takes into account all the features of decision maker's risks. If the parameter y of the aforementioned risk is known, then $H(X|y)$ is the premium for risk X whose characteristic is y. However, this specific feature y is usually a realization of some random variable Y. Therefore, the collective premium cannot be determined in a similar straightforward way, but it should be calculated in two steps. Firstly, an insurance company should determine $H(X|Y)$, which is a random variable dependent on Y. Then, a risk structure Y should be compensated by evaluating $H(H(X|Y))$. Since the premium $H(X)$ is in most cases different from $H(H(X|Y))$, there appears a problem to find under which circumstances these two values are the same.

Our main result of this paper is as follows:

Theorem 1. (i) *If $g(p) = h(p) = p$ and $u(x) = cx$, $u(x) = \left(1-e^{-cx}\right)/a$ or $u(x) = \left(e^{cx}-1\right)/a$ for $x \in \mathbb{R}$ and some $a, c > 0$, then $H(X)$ which is the solution of (2) is iterative.*

(ii) *Let u be a strictly increasing value function such that for all $x \in \mathbb{R}$ there exists the right-sided derivative of u which is finite and greater than 0 for all $x \neq 0$. Let $g, h \in \mathcal{G}$ be strictly increasing and continuous on $[0, 1]$ and suppose there exist finite one-sided derivatives $g'_-(x)$ and $h'_+(x)$ for $x \in (0, 1)$ and $0 < h'_+(0), g'_-(1) < \infty$. If the premium principle $H(X)$ is iterative for $w = 0$, then $g(p) = h(p) = p$ and $u(x) = cx$, $u(x) = 1-e^{-cx}$ or $u(x) = e^{cx}-1$ for all $x \in \mathbb{R}$ and some $a, c > 0$.*

The proof of Theorem 1 is given in Section 2. In this theorem we do not assume that functions u, g and h are differentiable which is consistent with the assumptions of Cumulative Prospect Theory.

Gerber (1974) proves that the premium principle which satisfies a continuity condition is iterative if and only if it is mean-value principle, i.e. it is the solution of $v(H(X)) = Ev(X)$, where v is a strictly increasing, convex and twice differentiable function. Under mild assumptions on value function and probability distortion functions, Theorem 1 states that the zero utility principle adjusted to Cumulative Prospect Theory is iterative if and only if the value function is linear or exponential and probability distortion functions are identities. It is known that if probabilities are not distorted, the zero utility principle with a linear or exponential value function is also the mean-value principle. Therefore, the main theorem of this paper is consistent with the result by Gerber (1974).

A generalization of the result by Gerber is given by Goovaerts and De Vylder (1979). They prove that the Swiss principle is iterative if and only if it reduces to mean-value principle or zero utility principle with a linear or exponential utility function. Gerber (1979) also notices that if $S = X_1 + \cdots + X_N$ is a random sum and premium principle $H(X)$ is both additive and iterative, then $H(S) = H(H(S|N)) = H(H(X) \cdot N)$. Moreover, Goovaerts et al. (2010) conclude that if the premium principle is a mixture of exponential functions, then it is iterative if and only if the mixture function is degenerate.

2. Proofs

Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Further, we write $f'(x)$ to denote the right-sided derivative of f at $x \in I \setminus \{\sup I\}$.

Lemma 2. *Let $f : [0, 1] \to \mathbb{R}$ be continuous. If f is right differentiable at every point $x \in [0, 1)$ and $f'(x) = 0$ for all $x \in [0, 1)$, then f is constant on $[0, 1]$.*

Remark 3. The proof of Lemma 2 is given by Rajwade and Bhandari (2007). The assumption on continuity of f cannot be omitted. In fact, the function $f(x) = \mathbf{1}_{[x_0,1]}(x)$, with $x_0 \in (0, 1]$, satisfies the equation $f'(x) = 0$ for all $x \in [0, 1)$ but f is not constant on $[0, 1]$.

Remark 4. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f(0) = a$, where $a \in \mathbb{R}$ is fixed. Suppose that f satisfies the differential equation $f'(x) = g(x)$ for all $x \in [0, 1)$, where function g is known. If G is a continuous solution of this differential equation, then $(f-G)' = 0$. From Lemma 2 it follows that G is the unique solution of the equation $f'(x) = g(x)$.