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# Constant proportion portfolio insurance under a regime switching exponential Lévy process

Chengguo Weng *

*Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, N2L 3G1, Canada*

## HIGHLIGHTS

- The CPPI portfolio is studied with price jumps and market regime switching.
- Results are derived under general Markov's regime switching Lévy models.
- Specific implementation is discussed under many popular Lévy models.
- Our results highlight the effects from the market state at the inception.

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## ABSTRACT

The constant proportion portfolio insurance is analyzed by assuming that the risky asset price follows a regime switching exponential Lévy process. Analytical forms of the shortfall probability, expected shortfall and expected gain are derived. The characteristic function of the gap risk is also obtained for further exploration on its distribution. The specific implementation is discussed under some popular Lévy models including the Merton's jump–diffusion, Kou's jump–diffusion, variance gamma and normal inverse Gaussian models. Finally, a numerical example is presented to demonstrate the implication of the established results.



## 1. Introduction

Portfolio insurance refers to those managing techniques designed to protect the value of a portfolio. They usually target to provide a guarantee on the terminal portfolio value by maintaining the portfolio value process not falling below a preset lower bound, which is called the *floor*. These techniques allow the investors to participate in equity market for its potential gains from an upside market move while limit the downside risk. The most prominent examples among the portfolio insurance strategies are the *constant proportion portfolio insurance* (CPPI) strategy and *option-based portfolio insurance* (OBPI) strategy. The OBPI combines a position in the risky asset with a put option on this asset; see for example El Karoui et al. (2005) and Leland and Rurbinstein (1988) among many others.

The CPPI strategy involves no option. It adopts a simplified self-financing strategy to allocate capital between a risky asset (typically a traded fund or index) and a reserve asset (typically a bond) dynamically over time. In this method, the investor starts by setting a *floor* equal to the lowest acceptable value of the portfolio. Then, the investor computes the *cushion* as the excess of the portfolio value over the floor and allocates in the risky asset an amount of a constant multiple of the cushion. The constant is called *multiplier*. The amount allocated to the risky asset is known as the *exposure*, and the remainders are all invested in the reserve asset.

* Tel.: +1 519 888 4567x31132.
*E-mail address:* c2weng@uwaterloo.ca.

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The CPPI strategy was initially introduced by Perold (1986) (see also Perold and Sharpe, 1988) for fixed-income instruments and Black and Jones (1987) for equity instruments. Extensive research has been conducted on CPPI in recent years, often either embedded in a more general framework or compared with other portfolio insurance strategies. A comparison of OBPI and CPPI (in continuous time) is given in Bertrand and Prigent (2005) and Balder and Mahayni (2010); also see Do (2002) for an empirical investigation of both methods via simulation using Australian data. The performance of credit CPPI and constant proportion debt obligation structures is studied by Garcia et al. (2008) under a dynamic multivariate jump-driven model for credit spreads, and an investigation in much more depth under a similar setting can be found in Joossens and Schoutens (2010). The effect from price jumps on the performance of the CPPI strategy is studied by Cont and Tankov (2009) under a general exponential Lévy process. The literature also deals with stochastic volatility models and extreme value approaches on the CPPI method; see Bertrand and Prigent (2002, 2003). A general framework of CPPI for investment and protection strategies is formulated by Dersch (2010) along with a review on other portfolio insurance techniques. The influence of estimation risk on the performance of CPPI strategies as well as the mitigation effect of the estimation risk by the robustification of mean–variance efficient portfolios is studied by Schöttle and Werner (2010). The effectiveness of a CPPI portfolio with proportional trading cost is investigated by Balder et al. (2009) under the Black–Scholes model and extended by Weng and Xie (2013) to a general exponential Lévy model. Moreover, a log-normal approximation approach for the gain of CPPI structure is also developed by Weng and Xie (2013).

When the trajectories of the price processes of both risky asset and reserve asset are continuous, the CPPI strategy with continuous trading will lead to a terminal portfolio value no less than the guaranteed value certainly, and hence fully achieve the purpose of portfolio insurance. Nevertheless, it has been widely noticed that there is always possibility for a CPPI portfolio to fall below the floor, leading to the notorious *gap risk*, which happens when the price of the risky asset drops substantially before the portfolio manager can rebalance the portfolio. Obviously, there are two main factors that may contribute to the gap risk: the illiquidity of the investment assets and the jump in the asset price. In this paper, we focus on the effect from the jump features of the asset price while presume that the investment assets are perfectly liquid.

The present paper is motivated by Cont and Tankov (2009), where, while the main results are derived under the exponential Lévy model assumption, the authors started with a semimartingale model setup and developed a general framework for evaluating the CPPI portfolio. The present paper aims to generalize the main results of Cont and Tankov (2009) to a regime switching exponential Lévy model. While the exponential Lévy process can well capture the jump feature in the price of financial assets, one of its obvious criticisms is its time homogeneity. In reality, the economic state usually shows an obvious feature of transition between two or among several states, and the financial return has quite different characteristics under a different economic state. As such, the regime switching model has been utilized widely nowadays; for its application in finance and actuarial science, see, for example, Buffington and Elliott (2002), Elliott et al. (1995, 2005), Hardy (2001), Li et al. (2008) and Siu (2005) among many others.

More specifically, in the present paper the dynamics of the asset prices are assumed to be governed by distinct exponential Lévy processes under different market states (e.g., bull and bear), and the transition from one market state to another is supposed to follow a hidden Markov process. The present paper obtains analytical forms for those important risk measures that are associated with the CPPI portfolio such as the short probability and the expected shortfall. The characteristic function of the shortfall is also obtained in an explicit form so as to make it possible to further explore on its distribution. In reality, the guarantee for the investors is usually provided by a bank (guarantor), which owns the CPPI portfolio, subject to a premium, and thus the gap risk is indeed assumed by the bank in exchange for the premium charged on the investors. Our established results will be helpful not only for the guarantor to conduct an effective evaluation on the gap risk and compute a reasonable level of premium but also for the investors to develop a good understanding on their risk-and-reward profile in investing a CPPI fund; for details, see the beginning part of Section 3.1. The specific implementation of our main results (and its challenges for some models) is investigated for some popular exponential Lévy processes including the Merton's jump–diffusion, Kou's jump–diffusion, variance gamma and normal inverse Gaussian models; see Section 4. Finally, our numerical example presented in Section 5 shows that the initial market state (bull or bear) of the investment will take a critical role in the resulting risk associated with a CPPI portfolio.

The rest of the paper is organized as follows. Section 2 is the model setup. The main results are collected in Section 3, following some preliminaries in the beginning of the section and followed by a discussion on how to derive the explicit formulas for the established main results. Section 4 investigates how to specifically implement main results for some popular exponential Lévy processes including the Merton's jump–diffusion, Kou's jump–diffusion, variance gamma and normal inverse Gaussian models. Section 5 presents a numerical example, where the effect of the regime switching feature of the market is demonstrated. Section 6 concludes the paper. Finally, proof of some lemmas and equations are relegated to the Appendix.

## 2. Model setup

Throughout the paper, we suppose that all the random elements involved are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that the expectation of a random variable $Z$ under $\mathbb{P}$ is denoted by $\mathbb{E}(Z)$. The transpose of a matrix $\mathbf{H}$ (a vector $\mathbf{a}$) will be denoted by $\mathbf{H}'$ ($\mathbf{a}'$).

Assume that the state of the financial market is described by a finite state Markov process $X := (X_t)_{t\geq 0}$. In particular, there could be just two states for $X$, respectively, representing *bull* and *bear*; if we assume a third market state, it is typically interpreted as a *normal* state. As introduced by Elliott et al. (1995) and Buffington and Elliott (2002), we suppose that the Markov process $X$ is generated by an intensity matrix $\mathbf{Q}$ with a finite state space of unit vectors $\{e_1, \ldots, e_n\}$, where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)' \in \mathbb{R}^n$ with 1 in its $k$th coordinate and 0 in all the others. According to Elliott et al. (1995), $X$ has the following semimartingale representation

$$X_s = X_0 + \int_0^s X_u \mathbf{Q}\, du + M_s, \tag{2.1}$$

where $M$ is a martingale with respect to the filtration $\mathcal{F}_t^X$ generated by $X$.

Hereafter we assume that the CPPI portfolio is allocated between a stock index and a zero-coupon bond, and their price processes $S$ and $B$ are respectively subject to the following dynamics

$$\frac{dS_t}{S_{t-}} = dZ_t \quad \text{and} \quad \frac{dB_t}{B_{t-}} = dR_t, \tag{2.2}$$

for two processes $Z$ and $R$ admitting the following regime switching structures

$$Z_t = \sum_{j=1}^n \langle e_j, X_t \rangle Z_t^{(j)}, \quad \text{and} \quad R_t = \sum_{j=1}^n \langle e_j, X_t \rangle R_t^{(j)}, \tag{2.3}$$