Insurance: Mathematics and Economics 52 (2013) 522–531

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

On the (in-)dependence between financial and actuarial risks

Jan Dhaene [a,*], Alexander Kukush [b], Elisa Luciano [c], Wim Schoutens [a], Ben Stassen [a]

[a] KU Leuven, Leuven, Belgium
[b] Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
[c] University of Torino - Collegio Carlo Alberto, Torino, Italy

HIGHLIGHTS

- We assume that under the real-world measure $\mathbb{P}$, financial and biometrical risks are mutually independent.
- One often makes the same assumption under the equivalent risk-neutral measure $\mathbb{Q}$.
- An asset price model where financial and biometrical risks are independent, might be impossible in an arbitrage-free market.
- We investigate the conditions to transfer the independence assumption from $\mathbb{P}$ to $\mathbb{Q}$.

ARTICLE INFO

Article history:
Received December 2012
Received in revised form
March 2013
Accepted 14 March 2013

Keywords:
Independence
Real-world probability measure $\mathbb{P}$
Risk-neutral probability measure $\mathbb{Q}$
Financial risks
Actuarial risks
Insurance securitization

ABSTRACT

Probability statements about future evolutions of financial and actuarial risks are expressed in terms of the 'real-world' probability measure $\mathbb{P}$, whereas in an arbitrage-free environment, the prices of these traded risks can be expressed in terms of an equivalent martingale measure $\mathbb{Q}$. The assumption of independence between financial and actuarial risks in the real world may be quite reasonable in many situations. Making such an independence assumption in the pricing world however, may be convenient but hard to understand from an intuitive point of view. In this pedagogical paper, we investigate the conditions under which it is possible (or not) to transfer the independence assumption from $\mathbb{P}$ to $\mathbb{Q}$. In particular, we show that an independence relation that is observed in the $\mathbb{P}$-world can often not be maintained in the $\mathbb{Q}$-world.

1. Introduction

'Insurance securitization' can be defined as the transfer of underwriting risk of the insurance industry to investors in capital markets through the issuance of financial securities of which the payoffs depend on the outcome of quantities related to this underwriting risk, see e.g. Gorvett (1999). Examples of such financial securities are longevity bonds and catastrophe bonds. Modeling and pricing these insurance-related instruments involve both financial and actuarial considerations. In this note, we will investigate the assumption of independence between pure financial and pure actuarial risks that is often made in this context. In particular, we will focus on the differences between this independence assumption when it is made in the physical world versus the pricing world. As this note is of a pedagogical nature, it is to a large extent written in a self-contained way.

As usual, we model the financial world with the help of a filtered probability space. Instantaneous interest rates and stock prices are stochastic processes adapted to the filtration in this probability space. Actuarial risks are described via adapted stochastic processes in a second filtered probability space. Hereafter, we will restrict actuarial risks to biometrical risks, such as remaining lifetimes of individuals or survival indices of populations, but our findings can immediately be applied to other actuarial risks as well, such as catastrophic loss indices. The combined financial and biometrical world is described via the product space of the two above-mentioned filtered measurable spaces. Real-world probabilities in this combined world are described by a measure $\mathbb{P}$, of which the projections to the financial and the biometrical subworlds coincide with the respective probability measures attached to these subworlds. Notice that in general the measure $\mathbb{P}$ is not the product of the measures attached to the subworlds, meaning that stochastic processes in the financial and in the biometrical world are not necessarily mutually independent.

* Corresponding author. Tel.: +32 473836093.
E-mail addresses: Jan.Dhaene@econ.kuleuven.be (J. Dhaene), Alexander_Kukush@univ.kiev.ua (A. Kukush), elisa.luciano@unito.it (E. Luciano), Wim.Schoutens@wis.kuleuven.be (W. Schoutens), Ben.Stassen@econ.kuleuven.be (B. Stassen).

0167-6687/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.insmatheco.2013.03.003

We assume a perfectly liquid and frictionless (no transaction costs, no trading constraints) market, as well as an arbitrage-free pricing framework. In this case, the physical probability measure $\mathbb{P}$ in the product space under consideration goes along with the existence of a (not necessarily unique) equivalent martingale measure $\mathbb{Q}$. Prices of exchange traded financial–biometrical risks are then given by discounted expectations, where expectations are taken with respect to $\mathbb{Q}$.

Hereafter, we will always assume that under the real-world measure $\mathbb{P}$, the dynamics of financial risks and biometrical risks are mutually independent, unless explicitly stated otherwise. This independence assumption may be quite reasonable and also intuitive in many cases. In the literature, one often makes the assumption that under the equivalent martingale measure $\mathbb{Q}$, the dynamics of financial risks and biometrical risks are also mutually independent. The latter assumption is very convenient as it allows us to separate the pricing of biometrical risk from the pricing of financial risk, but the intuitive idea behind this assumption is hard, if not impossible, to explain. In this paper, we focus on the meaning of independence in the pricing world. In particular, we investigate whether there is any relation between $\mathbb{P}$-independence and $\mathbb{Q}$-independence.

The remainder of this paper is structured as follows. In Section 2, we consider a combined financial–biometrical world with two possible scenarios in every subworld. In this simple world, we investigate the independence property of financial and biometrical risks by considering several examples. We start with a market which is home to traded assets of which the payoffs only depend on the outcome of one of both subworlds. In this incomplete world, an assumed independence which holds between financial and biometrical risks under the real-world probability measure $\mathbb{P}$ does not necessarily lead to an independence under the pricing measure $\mathbb{Q}$ that is chosen by the market. Here, 'chosen by the market' means that it follows implicitly from the prices of traded assets. Next, we complete this market by adding a combined financial–biometrical security. We show that, depending on the current price of the combined asset, it may be possible or not to find a pricing measure $\mathbb{Q}$ under which financial and biometrical risks are mutually independent. In order to prove that the non-existence of such a pricing measure is not related to the completeness of the market, we end Section 2 with an example of a combined incomplete market where it is impossible to find a pricing measure for which the independence property holds. In Section 3, we consider a general continuous-time combined financial–biometrical world and analyze pricing of traded mortality-linked derivatives, of which the payoffs depend on financial and biometrical evolutions. We investigate the relation between $\mathbb{P}$- and $\mathbb{Q}$-world independence among financial and biometrical risks. In Section 4, we consider an arbitrage-free bivariate Black & Scholes model. We show that under this model, independence relations between asset prices can be translated from the real world to the pricing world, and vice versa. Section 5 concludes the paper.

2. A simple combined financial–biometrical world

2.1. Financial and biometrical risks

In this section, we consider a combined financial–biometrical world in a discrete single period setting. This world is called 'combined' as it is hosting pure financial risks (such as stocks), as well as pure biometrical risks (such as a survival index related to a given population). Some (combinations) of the risks encountered in the combined world are traded (bought and sold) in a market. Throughout, we will assume that the market of these traded risks is arbitrage-free. Several of the observations that we will make concerning the theoretical example explored in this section will be formalized in a more realistic setting in Section 3.

Consider a *financial world* $\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)}\right)$, containing a risk-free bank account with interest rate equal to 0 (for notational and computational convenience) and a traded stock with initial price $S^{(1)}(0) = 100$. Eventual dividend payments can only occur at time 1. The price (cum dividend) of the stock in 1 years time is equal to either $S^{(1)}(1) = 50$ or $S^{(1)}(1) = 150$. The financial universe $\Omega^{(1)}$, which describes all possible evolutions of the financial world, is given by

$$\Omega^{(1)} = \{50, 150\},$$

where the different elements stand for the different possible values of the stock price at time 1. The σ-algebra $\mathcal{F}^{(1)}$ is the set of all subsets of $\Omega^{(1)}$. The elements of $\mathcal{F}^{(1)}$ are the events which may or may not occur in the financial world in the time interval $[0, 1]$. The probability measure $\mathbb{P}^{(1)}$, which attaches the 'real-world' probability to any event in $\mathcal{F}^{(1)}$, is characterized by the positive real numbers $\mathbb{P}^{(1)}[\{50\}] > 0$ and $\mathbb{P}^{(1)}[\{150\}] = 1 - \mathbb{P}^{(1)}[\{50\}] > 0$.

In the *biometrical world* $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)}\right)$, we observe a *survival index* which gives information about the survival experience of a given population, e.g. the population consisting of all persons in a given country. For simplicity, let us assume that the index $I(1)$ equals 0 in case 'few' persons survive during the experience year $[0, 1]$, whereas $I(1)$ equals 1 in case 'many' persons survive this year. The biometrical universe $\Omega^{(2)}$ describes all possible biometrical evolutions:

$$\Omega^{(2)} = \{0, 1\},$$

where the different elements stand for different values of the biometrical index at time 1. The σ-algebra $\mathcal{F}^{(2)}$ is the set of all subsets of $\Omega^{(2)}$. The elements of $\mathcal{F}^{(2)}$ are the events which may or may not occur in the biometrical world in the time interval $[0, 1]$. The probability measure $\mathbb{P}^{(2)}$ which attaches the 'real-world' probability to any event in the biometrical world, is characterized by the positive real numbers $\mathbb{P}^{(2)}[\{0\}]$ and $\mathbb{P}^{(2)}[\{1\}] = 1 - \mathbb{P}^{(2)}[\{0\}]$.

Next, we consider the *combined financial–biometrical world* $(\Omega, \mathcal{F}, \mathbb{P})$ which is the Cartesian product of the financial and the biometrical world. The universe Ω, generated by elements of the form (ω_1, ω_2) with $\omega_1 \in \Omega^{(1)}$ and $\omega_2 \in \Omega^{(2)}$, is given by

$$\Omega = \Omega^{(1)} \times \Omega^{(2)} = \{(50, 0), (150, 0), (50, 1), (150, 1)\}.$$

The σ-algebra $\mathcal{F}$ is the set of all events in the combined world. It is the set of all subsets of Ω:

$$\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} = \sigma\left(A \times B \mid A \in \mathcal{F}^{(1)}, B \in \mathcal{F}^{(2)}\right).$$

The probability measure $\mathbb{P}$ attaches the 'real-world' probability to any event in the combined world. Throughout this section, we will assume that financial and biometrical risks are independent in the following sense:

$$\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}, \tag{1}$$

where $\mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$ is the probability measure defined by

$$\mathbb{P}[\{\omega_1, \omega_2\}] = \mathbb{P}^{(1)}[\{\omega_1\}] \times \mathbb{P}^{(2)}[\{\omega_2\}], \quad \text{for any } \{\omega_1, \omega_2\} \in \mathcal{F}. \tag{2}$$

For ease of notation, hereafter we will denote $\mathbb{P}[\{\omega_1, \omega_2\}]$ as $\mathbb{P}[\omega_1, \omega_2]$, and $\mathbb{P}[\{\omega_i\}]$ as $\mathbb{P}[\omega_i]$, $i = 1, 2$. The independence assumption (2) immediately leads to

$$\begin{cases} \mathbb{P}[50, 0] = \mathbb{P}^{(1)}[50] \times \mathbb{P}^{(2)}[0] \\ \mathbb{P}[150, 0] = \mathbb{P}^{(1)}[150] \times \mathbb{P}^{(2)}[0] \\ \mathbb{P}[50, 1] = \mathbb{P}^{(1)}[50] \times \mathbb{P}^{(2)}[1] \\ \mathbb{P}[150, 1] = \mathbb{P}^{(1)}[150] \times \mathbb{P}^{(2)}[1]. \end{cases}$$