Insurance: Mathematics and Economics 52 (2013) 542–549

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Lifetime dependence modelling using a truncated multivariate gamma distribution

Daniel H. Alai *, Zinoviy Landsman, Michael Sherris

CEPAR, Risk and Actuarial Studies, Australian School of Business, UNSW, Sydney NSW 2052, Australia
Department of Statistics, University of Haifa, Mount Carmel, Haifa 31905, Israel

HIGHLIGHTS

- We model dependence within a group of lives using a multivariate gamma distribution.
- Model calibration is based on the method of moments and developed for truncated observations.
- The impact of dependence is demonstrated by applying the model to annuity valuation.
- Evidence is provided that confirms the relevance of dependent lifetimes.

ARTICLE INFO

Article history:
Received November 2012
Received in revised form February 2013
Accepted 24 March 2013

JEL classification:
G22
G32
C13
C02

Keywords:
Systematic longevity risk
Dependence
Multivariate gamma
Lifetime distribution
Annuity valuation

ABSTRACT

Systematic improvements in mortality increases dependence in the survival distributions of insured lives, which is not accounted for in standard life tables and actuarial models used for annuity pricing and reserving. Systematic longevity risk also undermines the law of large numbers, a law that is relied on in the risk management of life insurance and annuity portfolios. This paper applies a multivariate gamma distribution to incorporate dependence. Lifetimes are modelled using a truncated multivariate gamma distribution that induces dependence through a shared gamma distributed component. Model parameter estimation is developed based on the method of moments and generalized to allow for truncated observations. The impact of dependence within a portfolio, or cohort, of lives with similar risk characteristics is demonstrated by applying the model to annuity valuation. Dependence is shown to have a significant impact on the risk of the annuity portfolio as compared with traditional actuarial methods that implicitly assume independent lifetimes.

1. Introduction

Systematic improvements in mortality increases dependence in the survival distributions of insured lives, which is not accounted for in standard life tables and actuarial models used for annuity pricing and reserving. Systematic longevity risk also undermines the law of large numbers; a law that is relied on in the risk management of life insurance and annuity portfolios. Given recent worldwide trends by employers towards the elimination of pension scheme liabilities, understanding systematic longevity risk is especially relevant for bulk annuity providers; see, e.g., Hull (2009).

This paper applies a multivariate gamma distribution to model dependent lifetimes within a pool of individuals. We make use of the following representation of the gamma density:

$$f(x) = \frac{\alpha^{\gamma}}{\Gamma(\gamma)} x^{\gamma-1} e^{-\alpha x}, \quad x > 0,$$

where γ is the shape parameter and α is the rate parameter. Lifetimes are often modelled with parametric distributions such as the gamma distribution; see, e.g., Klein and Moeschberger (1997). Dependence between the lifetimes is captured with a common stochastic component. The multivariate dependence structure is developed from the trivariate reduction method used to generate two dependent random variables from three independent random variables. This trivariate method was used to generate the bivariate version of the multivariate gamma distribution in Chereiyan (1941). The method uses the fact that the sum of gamma random variables with the same rate parameter also follows a gamma

* Corresponding author at: CEPAR, Risk and Actuarial Studies, Australian School of Business, UNSW, Sydney NSW 2052, Australia. Tel.: +61 2 9385 7633.
E-mail addresses: daniel.alai@unsw.edu.au (D.H. Alai), landsman@stat.haifa.ac.il (Z. Landsman), m.sherris@unsw.edu.au (M. Sherris).

0167-6687/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.insmatheco.2013.03.011

distribution with that same rate parameter. The trivariate method was generalized to multivariate reduction and the bivariate gamma distribution model extended to the multivariate setting by Ramabhadran (1951) and applied by Mathai and Moschopoulus (1991), Chatelain et al. (2006), and others.

The paper develops estimation theory for a multivariate gamma distribution in the presence of truncation. To quantify the effect of dependence, life annuities are valued with the model and compared with valuation under the assumption of independent lifetimes. Given that the marginal distributions of lifetimes are unchanged when introducing dependence, the expected present value of the annuity payment streams are equivalent in the comparison. However, the variance is significantly larger when dependence is introduced. Risk-based capital reflects the variance of the payment stream, and the cost of this capital is reflected in the market pricing of annuities. Hence, we provide evidence that dependence is a significant factor with important implications for annuity pricing and risk-based capital. This agrees with previous investigations by Dhaene et al. (2000), Denuit et al. (2001), and others. The model presented here provides a tractable method for estimating the dependence and computing the distribution of life annuity values, a similar problem previously considered by, for example, Denuit (2008) and Dhaene and Denuit (2007). Finally, an assessment of the model fit to data is provided based on Norwegian population mortality. Some insight is provided on ways in which the fit can be improved, the implementation of which is anticipated in future research.

Organization of the paper. Section 2 defines the multivariate gamma dependence structure for survival models for a pool of lives. Section 3 provides the estimation of the parameters of the model by method of moments. We consider the case when samples are given both with and without truncation. The former is essentially more complicated, but is required in practice. The performance of the estimation methods is assessed by simulation. Section 4 outlines the application to survival theory including implications for annuity values and portfolio risk based on standard deviation of values. Section 5 reports the fitting of the model to Norwegian population data. Section 6 concludes the paper.

2. Multivariate gamma survival model

The model is applied to individuals within a pool of lives. We assume M pools of lives. The pools can, in general, be of individuals that share characteristics indicative of a common risk factor, for example, age. Let $T_{i,j}$ be the survival time of individual $i \in \{1, \dots, N_j\}$ in pool $j \in \{1, \dots, M\}$. Although the number of lives in each pool need not be identical, we make this assumption for simplicity and continue with $N_j = N$ for all j. We assume the following model for the individual lifetimes:

$$T_{i,j} = Y_{0,j} + Y_{i,j},$$

where

- $Y_{0,j}$ follows a gamma distribution with shape parameter γ_0 and rate parameter α_j, $G(\gamma_0, \alpha_j)$, $j \in \{1, \dots, M\}$,
- $Y_{i,j}$ follows a gamma distribution with shape parameter γ_j and rate parameter α_j, $G(\gamma_j, \alpha_j)$, $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$, and
- the $Y_{i,j}$ are independent, $i \in \{0, \dots, N\}$ and $j \in \{1, \dots, M\}$.

Hence, there is a common component $Y_{0,j}$ within each pool j that impacts the survival of the individuals of that pool (i.e., $Y_{0,j}$ captures the impact of systematic mortality dependence between the lives in pool j). The parameters γ_j and α_j can jointly be interpreted as the risk profile of pool j.

From the properties of the gamma distribution it immediately follows that the survival times $T_{i,j}$ are also gamma distributed with shape parameter $\tilde{\gamma}_j = \gamma_0 + \gamma_j$ and rate parameter α_j. One can see that, within each pool, individual lifetimes are dependent, and all follow the same gamma distribution, $G(\tilde{\gamma}_j, \alpha_j)$. A consequence of the model is that lives from different pools are mutually independent. The dependence considered in our model is pool specific. Although, in reality, any two lives are not strictly independent, it is reasonable to assume that certain groups of lives are more closely related due to common risk factors. Such lives would exhibit a higher level of dependence than others. Our model is effectively based on the assumption that the pools under consideration have much stronger common mortality risks than individuals more generally. Since the focus of our research is on the impact of dependence for a pool, the model provides an appropriate basis to assess this.

3. Parameter estimation

In this section, we consider parameter estimation using the method of moments. For an excellent reference we can suggest, for example, Lindgren (1993, Chapter 8, Theorem 6). Parameter estimation for the *bivariate* gamma distribution has been previously studied by Chatelain et al. (2006); they investigated both maximum likelihood and the method of moment estimation. For the *multivariate* gamma distribution, expressions of the probability density function can become rather challenging. For this reason, we establish our estimation procedure on the method of moments.

Notation

Before we undertake parameter estimation, we provide some necessary notation concerning raw and central, theoretical and sample, moments. Consider arbitrary random variable X. We denote with $\alpha_k(X)$ and $\mu_k(X)$ the kth, $k \in \mathbb{Z}^+$, raw and central (theoretical) moments of X, respectively. That is,

$$\alpha_k(X) = E[X^k],$$
$$\mu_k(X) = E[(X - \alpha_1(X))^k].$$

Next, consider random sample $\mathbf{X} = (X_1, \dots, X_n)'$. The raw sample moments are given by

$$a_k(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i^k, \quad k \in \mathbb{Z}^+.$$

For $X_1, \dots, X_n$ identically distributed, the raw sample moments are unbiased estimators of the corresponding raw moments of X_1:

$$E[a_k(\mathbf{X})] = \alpha_k(X_1).$$

Finally, we define the *adjusted* second and third central sample moments as

$$\widetilde{m}_2(\mathbf{X}) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - a_1(\mathbf{X}))^2,$$

$$\widetilde{m}_3(\mathbf{X}) = \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}(X_i - a_1(\mathbf{X}))^3.$$

For $X_1, \dots, X_n$ independent and identically distributed, these (adjusted) central sample moments are unbiased and consistent estimators of the corresponding central moments of X_1:

$$E[\widetilde{m}_2(\mathbf{X})] = \mu_2(X_1) \quad \text{and} \quad E[\widetilde{m}_3(\mathbf{X})] = \mu_3(X_1).$$

3.1. Parameter estimation for lifetime observations

We assume we are given samples, $\mathbf{T}_1, \dots, \mathbf{T}_M$, from the pools, where $\mathbf{T}_j = (T_{1,j}, \dots, T_{N,j})'$. This assumption requires the data used for calibration to be based on observed lifetimes for past lives. We allow for truncation, which is addressed in the paper, and leave allowance for censoring for future research.