Insurance: Mathematics and Economics 52 (2013) 560–572

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Multivariate distribution defined with Farlie–Gumbel–Morgenstern copula and mixed Erlang marginals: Aggregation and capital allocation

Hélène Cossette, Marie-Pier Côté, Etienne Marceau *, Khouzeima Moutanabbir

École d'Actuariat, Université Laval, Québec, Canada

HIGHLIGHTS

- We consider a portfolio of dependent risks.
- The joint distribution is defined with the FGM copula.
- Mixed Erlang distributions are assumed for the marginals.
- We show that the aggregate claim amount has a mixed Erlang distribution.
- The contributions of each risk are derived using the TVaR and covariance rules.

ARTICLE INFO

Article history:
Received July 2012
Received in revised form
February 2013
Accepted 26 March 2013

Keywords:
Aggregate claim loss
Risk measures
Capital allocation
Tail-Value-at-Risk
FGM copula
TVaR-based allocation rule
Covariance-based allocation rule
Mixed Erlang distribution

ABSTRACT

In this paper, we investigate risk aggregation and capital allocation problems for a portfolio of possibly dependent risks whose multivariate distribution is defined with the Farlie–Gumbel–Morgenstern copula and mixed Erlang distribution marginals. In such a context, we first show that the aggregate claim amount has a mixed Erlang distribution. Based on a top-down approach, closed-form expressions for the contribution of each risk are derived using the TVaR and covariance rules. These findings are illustrated with numerical examples.

1. Introduction

In light of the new regulation requirements, insurance companies are required to determine their capital allocation according to their risk exposure. In such a context, risk management raises some issues about risk aggregation and capital allocation. A risk capital must be held by the institution for the whole business portfolio to insure a safety financial level and also to be allocated adequately to each risk. Required capitals are commonly determined using an adequate risk measure, which is a mapping from the random variable space into the real numbers that allows risk ordering. Artzner et al. (1999) give an axiomatic definition of a risk measure and introduce the concept of coherent measures of risk. Artzner (1999) examines the implication of using coherent risk measures on capital requirements in an insurance context. As for Wang (2002), he notably discusses coherent methods to determine the aggregate capital requirement for a firm and the capital allocation to individual business units. These methods for enterprise risk management can be used for asset/loss portfolio optimization. Both Artzner (1999) and Wang (2002) suggest using the Tail Value at Risk (TVaR), also called the Expected Shortfall (ES), to replace the usual Value at Risk given that it does not meet the subadditivity criterion. The TVaR is a coherent risk measure and it is equal to the Conditional Tail Expectation (CTE) in the continuous case. In a discrete setting, as explained in Acerbi et al. (2001) and Acerbi and Tasche (2002), the TVaR remains a coherent risk measure while the CTE is no longer coherent. See also McNeil et al. (2005) for details on risk measures

* Corresponding author.
E-mail address: etienne.marceau@act.ulaval.ca (E. Marceau).

0167-6687/$ – see front matter
http://dx.doi.org/10.1016/j.insmatheco.2013.03.006

and their applications in a quantitative risk management context. For a recent discussion and further details on the use and origin of the VaR and TVaR measures of risk, and the families they belong too, the reader is referred to Goovaerts et al. (2010).

To allocate capital to different lines of business, Denault (2001) suggests a set of desirable properties for a fair risk capital allocation principle. More precisely, his axioms define the coherence of risk capital allocation principles, in a similar way as Artzner et al. (1999) in the context of risk measures. The top down allocation method introduced by Tasche (2000) has been used to provide several closed-form formulas and approximations of the TVaR and the TVaR-based allocations for different types of multivariate continuous distributions. For example, Panjer (2002) shows that the TVaR-based allocation principle is identical to the covariance-based principle when multivariate normal distributions are considered. Dhaene et al. (2008) develop a closed-form expression for the TVaR allocation under multivariate elliptical distributions. Bargès et al. (2009) give a closed-form expression for the TVaR-based allocation when lines of business of an insurance portfolio are linked with a Farlie–Gumbel–Morgenstern (FGM) copula and when marginal risks are distributed as mixtures of exponentials. Other applications of the TVaR-based allocation principle are also provided in Cossette et al. (2012). Buch and Dorfleitner (2008) discuss the gradient allocation principle which generalizes well known allocation principles including the TVaR and covariance rules. The risk aggregation problem using VaR and TVaR, and the key differences between these two measures of risk, have been studied thoroughly in Kaas et al. (2009).

In this paper, we address risk aggregation and capital allocation problems for a portfolio of dependent risks whose multivariate distribution is defined with a copula and mixed Erlang distributed marginals. The class of mixed Erlang distributions has many interesting features which are discussed in detail notably in Willmot and Lin (2010) (see also references therein). With several examples, they illustrate the versatility and the usefulness of this class of distributions for modeling claim amounts and the availability of closed-form expressions for various quantities of interest in risk theory. Any positive continuous distribution may be approximated by a member of the mixed Erlang class which includes distributions such as the generalized Erlang distribution and phase-type distributions. Furthermore, the mixed Erlang class provides many possible shapes of probability density functions (pdf) and a variety of skewness in the right tail. These features make this class very useful in actuarial and risk management applications. Lee and Lin (2010) suggest the use of mixed Erlang distributions to model insurance losses. For the dependence structure, we use the FGM copula which is attractive due to its simplicity and hence allows explicit results. This copula is a perturbation of the independence copula. In this paper, we capitalize on the ability to write a joint pdf defined by an FGM copula in terms of the product of marginal pdfs and their corresponding survival functions. We combine the tractability of the FGM copula and the mixed Erlang distribution class to analyze the stochastic behavior of the aggregate claim amount for a portfolio of dependent risks in order to determine the amount of economic capital needed for the whole portfolio. More precisely, we show under these assumptions that the aggregate claim amount follows a mixed Erlang distribution. Then, we find closed-form expressions for the corresponding TVaR risk measure and stop-loss premium. Based on a top-down approach, explicit expressions for the amount of capital to be allocated to each risk based on the TVaR and covariance rules are derived.

The paper is organized as follows. Section 2 fixes some notations, definitions and provides some preliminary results. The expressions for the bivariate case are given in Section 3 together with essential theorems and an illustrative example. Section 4 shows how to extend the results to the multivariate case and provides a numerical example for the trivariate case.

2. Definitions

In this section, we briefly recall the definition and characteristics of the FGM copula. We also give the definitions of the risk measures Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) as well as the allocation rules based on the TVaR and the covariance (see e.g. McNeil et al., 2005). We end this section by presenting the mixed Erlang distribution and its properties.

2.1. Farlie–Gumbel–Morgenstern copula

Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n continuous random variables (rvs) with joint cumulative distribution function (cdf) denoted by $F_{\underline{X}}$ and univariate marginals F_{X_i}, $i = 1, \ldots, n$. According to Sklar's theorem, see e.g. Sklar (1959) and Nelsen (2006), $F_{\underline{X}}$ can be written as a function of the univariate marginals F_{X_i}, $i = 1, \ldots, n$, and the copula C describing the dependence structure as follows:

$$F_{\underline{X}}(x_1, \ldots, x_n) = C\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right).$$

The joint probability density function (pdf) of $\underline{X}$ is given by

$$f_{\underline{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \ldots f_{X_n}(x_n)\, c\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right), \quad (1)$$

where c is the corresponding pdf of the copula C defined by

$$c(u_1, \ldots, u_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}.$$

In this paper, we are interested in the FGM copula. The bivariate FGM copula is defined by the joint cdf

$$C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \quad (2)$$

where the scalar θ is the dependence parameter with $\theta \in [-1, 1]$. The independence structure is reached when $\theta = 0$, i.e. $C^I(u_1, u_2) = u_1 u_2$. The pdf of the bivariate FGM copula is given by

$$c(u_1, u_2) = (1 + \theta) - \theta 2\overline{u}_1 - \theta 2\overline{u}_2 + \theta 2\overline{u}_1 2\overline{u}_2, \quad (3)$$

where $\overline{u}_i = 1 - u_i$.

The FGM copula is a perturbation of the product copula and, as mentioned in Nelsen (2006), is a first order approximation to the Ali Mikhail Haq, Frank and Placket copulas. With $\theta \in [-1, 1]$ and association measures such as Kendall's tau and Spearman's rho respectively given by $\tau = \frac{2\theta}{9}$ and $\rho = \frac{\theta}{3}$, moderate positive and negative dependence can be modeled with the FGM copula. This copula is attractive due to its simplicity and its form which allows explicit calculus and exact results. For example, Bargès et al. (2009) investigate aggregation and capital allocation problems for an insurance company with several lines of business with dependence structure based on the FGM copula and with exponentially distributed risks. Prieger (2002) highlights its usefulness in model selection into health insurance plans. The FGM copula was also used to link claim variables in a credibility model in Yeo and Valdez (2006). The FGM copula with exponential margins was proposed by Jang and Fu (2011) to measure tail dependence between collateral losses. In finance, Cherubini et al. (2011) use the FGM copula for the analysis of financial time series and suggest a new technique to construct first order Markov processes using this copula. The FGM copula was also used to describe different correlation relations on the financial markets in Gatfaoui (2005, 2007). In risk theory, Cossette et al. (2008), Zhang and Yang (2011) and Chadjiconstantinidis and Vrontos (2012) consider risk models with a dependence structure between claim sizes and interclaim times based on the FGM copula.