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Pricing compound Poisson processes with the Farlie–Gumbel–Morgenstern dependence structure

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ABSTRACT

Convenient expressions for the Esscher pricing functional in the context of the compound Poisson processes with dependent loss amounts and loss inter-arrival times are developed. To this end, the moment generating function of the aforementioned dependent processes is derived and studied. Various implications of the dependence are discussed and exemplified numerically.

1. Introduction

Let $\mathcal{X}$ denote a set of actuarial losses, and assume that these losses are represented by non-negative random variables (r.v.'s), say $\mathcal{X} \ni X : \Omega \to [0, \infty) := \mathbf{R}_{0,+}$, defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. (We note in passing, that the r.v. X must not generally be a scalar-valued one.) In addition, let $\pi : \mathcal{X} \to [0, \infty] := \overline{\mathbf{R}}_{0,+}$ denote a pricing functional, assigning a monetary equivalence to each X in $\mathcal{X}$.

Numerous studies involving pricing functionals π are available in the literature. In this respect, properties of relevant objects have been studied and new ones have been proposed in, e.g., distribution theory (see, e.g.,Furman and Landsman, 2010; Constantinescu et al., 2011 and Yang et al., 2011, and references therein) and pricing theory (see, e.g., Young, 2004, for a review, and Wang, 1996; Furman and Zitikis, 2009, and references therein), among others. Intersections are also available (see, e.g., Furman and Landsman, 2006; Vernic, 2010 and Albrecher et al., 2011, and references therein).

However, a more intricate problem of pricing a stochastic process $X_t : \Omega \to \mathbf{R}_{0,+}$, with $t \in \mathcal{T} \subseteq \mathbf{R}_{0,+}$, is seemingly somewhat less well-studied, even though in essence actuaries face X_t rather than X in their practice. In view of the above, in this note we are concerned with a collection of stochastic processes $\mathfrak{X}_{\mathcal{T}}$ and a pricing functional $\Pi : \mathfrak{X}_{\mathcal{T}} \to \overline{\mathbf{R}}_{0,+}$, with $\Pi[X_t]$ denoting the price for $X_t \in \mathfrak{X}_{\mathcal{T}}$.

Motivated by the classic collective risk theory (see, Bühlmann, 1970), in the sequel we are interested in the processes of the form

$$X_t := Y_1 + Y_2 + \cdots + Y_{N_t} = \sum_{n=1}^{N_t} Y_n, \quad \text{for } N_t > 0, \tag{1.1}$$

where, for a sequence of loss occurrence times $\{T_n\}_{n\geq 1}$, we denote by $N_t := \sup\{n \in \mathbf{N} : T_n \leq t \in \mathcal{T}\}(N_0 = 0)$ a usual renewal process, and by $\{Y_n\}_{n\geq 1}$, a sequence of the corresponding loss amounts. Also, we let $\{W_n\}_{n\geq 1}$ be a sequence of loss inter-arrival times, with $W_n := T_n - T_{n-1},\ n \geq 2$, and $W_1 := T_1$.

The process X_t is of prime applied importance, and it quantitatively describes the aggregate loss, an insurer incurs during the period $[0, t]$, with $t \in \mathcal{T}$ being generally known. We note in passing, that in practice, a non-zero discounting of $\{Y_n\}_{n\geq 1}$ may be desirable, and it can indeed be introduced into (1.1), making this non-trivial object even less tractable. However, in the major part of this note, we shall assume that the inflation and interest rates are equal, and thus random sum (1.1) is satisfactory.

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Inconveniently, explicit formulas are seldom derivable when tackling pricing functionals Π (see, Asmussen and Albrecher, 2010). An important and tractable practical case occurs when the loss inter-arrival times are identically exponentially distributed as $W \backsim \text{Exp}(\lambda)$. In such a case, which is also the one of interest in this work, we readily have that $N_t \backsim \text{Poisson}(\lambda t)$, i.e., the number of losses in the interval $[0, t]$ is distributed Poisson with mean λt.

Furthermore, the aforementioned exponentiality of the loss inter-arrival times is not the only assumption being imposed on (1.1) in an attempt to increase its tractability. The seemingly most restrictive simplification in this respect, is the one requiring independence of the loss amounts and loss inter-arrival times. In this note we allow for a pairwise dependence between the sequences $\{W_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$, and we demonstrate that the resulting dependent compound Poisson process remains to an extent analytically convenient. Our main result is formulated and proved in Section 2, and it is then applied to the Esscher-based pricing in Section 3. The problem of non-zero discounting in the framework of the dependent compound Poisson processes is discussed in Section 4, which concludes the paper.

2. Dependent compound Poisson processes and their moment generating function

In this section, we explore the moment generating function (m.g.f.) of random sums (1.1). To this end, the following assumptions are imposed, and they hold throughout, if it is not stated otherwise:

A. the inter-arrival times $\{W_n\}_{n\geq 1}$ are independent and identically distributed (i.i.d.) as a canonical r.v. W,

B. the aforementioned r.v. W is exponentially distributed with mean $\mathbf{E}[W] = 1/\lambda$,

C. the loss amounts $\{Y_n\}_{n\geq 1}$ are i.i.d. having the cumulative distribution function (c.d.f.) F and probability density function (p.d.f.) f on $\mathbf{R}_{0,+}$, and such that the corresponding m.g.f. is finite, i.e., $M(h) := \mathbf{E}[e^{hY}] < \infty$, for $h \in \mathcal{H} \subseteq \mathbf{R}$, and

D. the sequence $\{W_n, Y_n\}_{n\geq 1}$ consists of mutually independent bivariate r.v.'s (W, Y), with the dependence structure between W and Y being described by the Farlie–Gumbel–Morgenstern (FGM) copula having the c.d.f.

$$C_\theta(u, v) := uv + \theta uv(1-u)(1-v), \quad \text{for } -1 \leq \theta \leq 1, \quad (2.1)$$

and for $(u \times v) \in [0, 1] \times [0, 1]$ (see., e.g., Nelsen, 2006 and Cossette et al., 2010).

Note 2.1. In the sequel to distinguish between dependent and independent compound Poisson processes, we index the former ones with θ and thus write $X_{t,\theta}$, reserving the X_t notation for the classic independent case.

In short, assumptions A and B readily yield $N_t \backsim \text{Poisson}(\lambda t)$, assumption C is required because we are interested in the Esscher pricing functional, and assumption D conveniently implies the FGM copula-based dependence between the loss amounts and loss inter-arrival times (see, Nelsen, 2006).

More specifically, for the random pair (W, Y), having the c.d.f. C_θ, Spearman's rho and Kendall's tau are given by $\theta/3$ and $2\theta/9$, respectively. Furthermore, Pearsons' correlation is not surprisingly dependent on the marginal distributions of W and Y, and it is given by

$$\mathbf{Corr}[W, Y] = \theta \int_0^\infty F_W(x)(1 - F_W(x))dx \times \int_0^\infty F_Y(x)(1 - F_Y(x))dx$$

(see, e.g., Mari and Kotz, 2001). Thus FGM copulas allow for both negative and positive dependences between the loss amounts and loss inter-arrival times. In addition, FGM copulas are first-order approximations to Ali–Mikhail–Haq, Frank and Placket copulas (see, Hutchinson and Lai, 1990 and Mari and Kotz, 2001, and references therein for additional properties). We depict $C_\theta(u, v)$ for various dependence levels in Fig. 1.

In the sequel, we make extensive use of the following definition (see, e.g., Furman and Zitikis, 2009, and references therein).

Definition 2.1. The random variable $Y_w \backsim F_w$, such that

$$F_w(y) := \frac{\mathbf{E}[\mathbf{1}\{Y \leq y\} w(Y)]}{\mathbf{E}[w(Y)]}, \quad y \in \mathbf{R},$$

where $0 < \mathbf{E}[w(Y)] < \infty$, is called the weighted variant of $Y \backsim F$ with the weight function $w : \mathbf{R} \to \mathbf{R}_+$.

Using Definition 2.1 and assumption D, we readily observe that the p.d.f. of (W, Y) is conveniently written as

$$f_{W,Y}(w, y) = \lambda e^{-\lambda w} f(y) + \theta(f(y) - f_w(y))(2\lambda e^{-2\lambda w} - \lambda e^{-\lambda w}), \quad (2.2)$$

where f_w is the p.d.f. of the weighted counterpart of Y with the weight function $w(y) = 2F(y)$ (note that $\mathbf{E}[2F(Y)] = 1$).

Curiously, certain differences involving $f(y)$ and $f_w(y)$ are of certain importance when analyzing the implications of our main result. Therefore, we find the following simple proposition useful.

Proposition 2.1. *Let $Y \backsim F$ and $Y_w \backsim F_w$, with $w(y) = 2F(y)$. Then*

$$M_w(h) := \mathbf{E}[e^{hY_w}] \geq M(h), \quad (2.3)$$

for all $h \in \mathcal{H} \subseteq \mathbf{R}$, such that the m.g.f.'s above exist.

Proof. Directly by definition

$$\begin{aligned}\int_0^\infty e^{hy} dF_w(y) &= \int_0^\infty \sum_{k=0}^\infty \frac{(hy)^k}{k!} dF_w(y) = \sum_{k=0}^\infty \frac{h^k}{k!} \mathbf{E}[Y_w^k] \\ &= \sum_{k=0}^\infty \frac{h^k}{k!} \mathbf{E}[Y^k 2F(Y)] \\ &\geq \sum_{k=0}^\infty \frac{h^k}{k!} \mathbf{E}[Y^k] = \sum_{k=0}^\infty \frac{h^k}{k!} \int_0^\infty y^k dF(y) \\ &= \int_0^\infty e^{hy} dF(y),\end{aligned}$$

with the inequality due to the fact that, for $k \in \mathbf{N}$, we have that Y^k and $2F(Y)$ are positive quadrant dependent (PQD) (see, Lehmann, 1966), and thus we have that

$$\mathbf{E}[Y^k(2F(Y))] \geq \mathbf{E}[Y^k]\mathbf{E}[(2F(Y))] = \mathbf{E}[Y^k],$$

which justifies the inequality and thus concludes the proof. □

In what follows we denote by $H^*(p)$ the Laplace transform of $H(x)$, for $x \geq 0$. Namely, we have that

$$H^*(p) := (\mathcal{L}H)(x) = \int_0^\infty e^{-px} H(x) dx.$$

The following theorem establishes the m.g.f. of the dependent compound Poisson process.

Theorem 2.1. *Let $X_{t,\theta}$ be a dependent compound Poisson process as described by assumptions* A–D. *Then its moment generating function is formulated as*

$$M_{X_{t,\theta}}(h) = \frac{(2\lambda + p_{1,\lambda,\theta}(h))e^{p_{1,\lambda,\theta}(h)\, t} - (2\lambda + p_{2,\lambda,\theta}(h))e^{p_{2,\lambda,\theta}(h)\, t}}{p_{1,\lambda,\theta}(h) - p_{2,\lambda,\theta}(h)}, \quad (2.4)$$