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Consumption, investment and life insurance strategies with heterogeneous discounting

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HIGHLIGHTS

- Time-inconsistent preferences.
- Consumption/investment and life insurance model with heterogeneous discounting.
- Agent's increasing concern towards legacy and retirement modeling.
- Time-consistent strategies for CRRA and CARA utility functions.

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ABSTRACT

In this paper we analyze how the optimal consumption, investment and life insurance rules are modified by the introduction of a class of time-inconsistent preferences. In particular, we account for the fact that an agent's preferences evolve along the planning horizon according to her increasing concern about the bequest left to her descendants and about her welfare at retirement. To this end, we consider a stochastic continuous time model with random terminal time for an agent with a known distribution of lifetime under heterogeneous discounting. In order to obtain the time-consistent solution, we solve a non-standard dynamic programming equation. For the case of CRRA and CARA utility functions we compare the explicit solutions for the time-inconsistent and the time-consistent agent. The results are illustrated numerically.

1. Introduction

The introduction of an uncertain lifetime in portfolio optimization models has proved to be useful in the study of the demand for life insurance, which has usually been derived from a bequest function. The starting point for modern research on the subject dates back to Yaari (1965) who studied the problem of life insurance in a deterministic financial environment with the stochastic time of death as the only source of uncertainty. Later on, Richard (1975) combined the portfolio optimization model in Merton (1969, 1971) with the model in Yaari (1965) to deal with a life-cycle consumption/investment problem in the presence of life insurance and random terminal time. However, the model introduced by Richard (1975) had several unsatisfactory aspects. First, the value function was not well-defined at the final time because the random variable used to model the lifetime was assumed to be bounded. This is a very important point in view of the fact that the problem was analyzed using a dynamic programming approach, which proceeds backward in time. Second, as Leung (1994) pointed out, there is a problem with the existence of interior solutions. In order to overcome these difficulties, Pliska and Ye (2007) incorporated the randomness of the planning horizon by means of the uncertain life model found in reliability theory. In contrast to Richard (1975), in which the random lifetime took values on a bounded interval,

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in that paper the authors considered an intertemporal model and allowed the random lifetime to take values on $[0, \infty)$. In addition, the authors refined the theory in the following ways. First, the planning horizon was considered to be some fixed point in the future T (the retirement time for the decision maker) in contrast with the model in Richard (1975) in which the planning horizon was interpreted as the finite upper bound on the lifetime. Second, at T a utility was introduced accounting for the agent wealth at the final time. After setting up the HJB equation and deriving the optimal feedback control law, Pliska and Ye (2007) obtained explicit solutions for the family of discounted CRRA utilities. As is customary in the analysis of intertemporal decision problems, the decision maker considered was characterized by a constant discount rate of time preference, i.e., she discounted the stream of utilities of any category using an exponential discount function with a constant discount rate of time preference according to the Discounted Utility (DU) model introduced in Samuelson (1937). Within this framework, the marginal rate of substitution between payments at different times depends only on the length of the time interval contemplated – this fact probably being the main limitation of the DU model with regard to its capacity to describe the actual time preference patterns.

In fact, the empirical findings on individual behavior seem to challenge some of the predictions of the standard discounting model (see Frederick et al., 2002 for a review of the literature until then). For this reason, variable rates of time preference have received increasing attention in recent years, in attempts to capture the reported anomalies. In this sense, for instance, several papers focused on the greater impatience of decision makers about the choices in the short run compared with those in the long term using the hyperbolic discount function introduced by Phelps and Pollak (1968). Along the same lines, Karp (2007) and Marín-Solano and Navas (2010) dealt with the problem with non-constant discounting. Also, in a recent paper by Ekeland et al. (2012), the model of Pliska and Ye (2007) was extended with the introduction of non-constant discount rates.

The choice of the discount function will depend, in general, on the problem under consideration. For instance, in intertemporal problems with a bequest motive, like those studying the demand for life insurance, it is useful to account for the fact that the agent concern about the bequest left to her descendants is not the same when she is young than when she is an adult. A similar effect could be considered in retirement and pension models, in which the willingness to save for a better retirement is likely to be greater at the end of the working life than at the beginning. In addition, for such a long planning horizon the greater impatience in the short run may still play a role, although this bias should evolve according to the different valuations over time of the bequest and the pension plan. In order to capture this asymmetric valuation Marín-Solano and Patxot (2012) introduced the heterogeneous discounting model. According to these authors, the individual preferences at time t take the form

$$\int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho(T-t)} F(x(T), T), \tag{1}$$

i.e., the agent uses a constant discount rate of time preference, but this rate is different for the instantaneous utilities $L(x(s), u(s), s)$ and for the final function $F(x(T), T)$ which, in the previous examples, would account for the bequest or the agent wealth at retirement. The most relevant effect of using any non-constant discount function is that preferences change with time. Impatient agents over-valuing instantaneous utilities in comparison with the final function are characterized by $\rho > \delta$ in Eq. (1). However, as we approach the end of the planning horizon T the relative value of the final function increases compared with the instantaneous utilities, and consequently the bias to the present decreases with time (see Marín-Solano and Patxot, 2012 and de-Paz et al., 2013 for a detailed discussion of this effect).

The aim of this paper is to derive the optimal consumption, investment and life insurance rules for an agent whose concern about both the bequest left to her descendants and her wealth at retirement, compared to consumption, increases with time. To this end we depart from the model in Pliska and Ye (2007) generalizing the individual time preferences by incorporating heterogeneous discount functions. In contrast to the extension of Pliska and Ye's (2007) model in Ekeland et al. (2012), where an intergenerational problem is introduced by assigning different discount functions to different generations, our setting of heterogeneous discounting focuses on the time preference dynamics of the decision maker, i.e., our setting faces an intragenerational problem. Finally, we analyze how the standard solutions are modified depending on the attitude of the agent towards her changing preferences, showing the differences with some numerical illustrations.

In effect, the individual facing the problem of maximizing (1) can act in two different ways. On the one hand, she could solve the problem by ignoring the fact that her preferences are going to change in the near future, and applying the classical HJB equation. In this case, the strategies obtained will be only optimal from the point of view of her preferences at time t and, in general, will be only obeyed at that time; therefore they are time-inconsistent. On the other hand, she could take into account her changing preferences and obtain the time-consistent strategies by calculating Markov Perfect Equilibria (MPE). These different solutions are usually referred to as naive (in general time-inconsistent) and sophisticated (time-consistent) in the non-constant discounting literature. In order to obtain the MPE, Marín-Solano and Patxot (2012) derived the Dynamic Programming Equation (DPE) in a deterministic framework following a variational approach. The extension to the stochastic case, in which the state dynamics is described by a set of diffusion equations of the form $dX(t) = f(X(t), u(t), t)\, dt + \sigma(X(t), u(t), t) dz(t)$, where $z(t)$ is a standard Wiener process, was studied in de-Paz et al. (2013). In that paper the DPE providing time-consistent solutions was derived following two different approaches. The first one consisted in obtaining the DPE for the heterogeneous discounting problem in discrete time and then taking the formal continuous time limit, following Karp (2007) for the non-constant discounting problem in a deterministic setting (see Marín-Solano and Navas, 2010 for the stochastic case). The second one was the variational approach, as in Marín-Solano and Patxot (2012) (which is based on Ekeland and Lazrak, 2010). It is important to remark that, despite the fact that the two approaches are different in nature, the equilibrium conditions coincide.

Therefore, according to de-Paz et al. (2013) we have the following proposition.

Proposition 1. *Let $V(x, t)$ be a function of class $C^{2,1}$ in (x, t) satisfying the DPE*

$$\begin{aligned} \rho V(x, t) - V_t(x, t) - K(x, t) &= \sup_{\{u\}} \Big\{ L(x, u, t) + V_x(x, t) f(x, u, t) \\ &\quad + \frac{1}{2}\mathrm{tr}(\sigma(x, u, t) \cdot \sigma'(x, u, t) \cdot V_{xx}(x, t)) \Big\}, \end{aligned} \tag{2}$$

with $V(x, T) = F(x, T)$ and

$$K(x, t) = (\rho - \delta) E \left[\int_t^T e^{-\delta(s-t)} L(X(s), \phi(X(s), s), s)\, ds \right]. \tag{3}$$

Then $V(x, t)$ is the value function of the time-consistent (sophisticated) agent for the problem of maximizing the expected value of (1) subject to the corresponding state equation. If, for each pair (x, t), there exists a decision rule $u^* = \phi(x, t)$, with corresponding