Insurance: Mathematics and Economics 54 (2014) 93–108

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Risk aggregation with dependence uncertainty

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HIGHLIGHTS

- Introduce the Admissible Class for modeling dependence uncertainty.
- Obtain convex ordering lower bounds for dimension >2.
- Connect the sharpness of convex ordering bounds with complete mixability.
- Find explicit bounds on some convex and coherent risk measures including TVaR.
- Relate VaR bounds and convex ordering bounds.

ARTICLE INFO

Article history:
Received August 2013
Received in revised form
November 2013
Accepted 9 November 2013

Keywords:
Dependence structure
Aggregate risk
Admissible risk
Convex risk measures
TVaR
Convex order
Complete mixability
VaR bounds

ABSTRACT

Risk aggregation with dependence uncertainty refers to the sum of individual risks with known marginal distributions and unspecified dependence structure. We introduce the admissible risk class to study risk aggregation with dependence uncertainty. The admissible risk class has some nice properties such as robustness, convexity, permutation invariance and affine invariance. We then derive a new convex ordering lower bound over this class and give a sufficient condition for this lower bound to be sharp in the case of identical marginal distributions. The results are used to identify extreme scenarios and calculate bounds on Value-at-Risk as well as on convex and coherent risk measures and other quantities of interest in finance and insurance. Numerical illustrations are provided for different settings and commonly-used distributions of risks.

1. Introduction

In quantitative risk management, *risk aggregation* refers to the (probabilistic) behavior of an aggregate position $S(\mathbf{X})$ associated with a risk vector $\mathbf{X} = (X_1, \dots, X_n)$, where $X_1, \dots, X_n$ are random variables representing individual risks (one-period losses or profits). In this paper, we focus on the most commonly studied aggregate risk position, that is the sum $S = X_1 + \cdots + X_n$, since it has important and self-explanatory financial implications as well as tractable probabilistic properties.

In practice, there exist efficient and accurate statistical techniques to estimate the respective marginal distributions of $X_1, \dots, X_n$. On the other hand, the joint dependence structure of $\mathbf{X}$ is often much more difficult to capture: there are computational and convergence issues with statistical inference of multi-dimensional data, and the choice of multivariate distributions is quite limited compared to the modeling of marginal distributions. However, an inappropriate dependence assumption can have important risk management consequences. For example, using the Gaussian multivariate copula can result in severely underestimating probability of simultaneous default in a large basket of firms (McNeil et al., 2005). In this paper, we focus on the case when the marginal distributions of $X_1, \dots, X_n$ are known and the dependence structure of $\mathbf{X}$ is unspecified. This scenario is referred to as *risk aggregation with dependence uncertainty*. To study the aggregate risk when the information of dependence is unavailable or unreliable, we introduce the concept of *admissible risk* as a possible aggregate risk S with given marginal distributions but unknown dependence structure.

We are particularly interested in the convex order of elements in an admissible risk class. Generally speaking, convex order is consistent with preferences among admissible risks for all

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http://dx.doi.org/10.1016/j.insmatheco.2013.11.005

risk-avoiding investors. Previous studies on the convex order of admissible risks mainly focused on the sharp upper bound for a general number n of individual risks and the sharp lower bound for $n = 2$ (for example, see Denuit et al., 1999, Tankov, 2011 and Bernard et al., 2012, 2013a). In this paper, however, we focus on the sharp lower bound when $n \geqslant 3$, which is known to be an open problem for a long time. We show that the existence of a convex ordering minimal element in an admissible risk class is not guaranteed by providing a counterexample, and give conditions under which it exists. One of the conditions involves checking complete mixability (Wang and Wang, 2011). In the last section we propose a numerical technique to check this property, which suggests that the Gamma and Log-Normal distributions are completely mixable.

As we will show, a convex ordering lower bound can be useful to quantify the model risk and in many financial applications. A first application is to quantify model risk in capital requirements. Regulators and companies are usually more concerned about a risk measure $\rho(S)$ (as a measure of risk exposure or as capital requirements needed to hold the position S over a pre-determined period) instead of the exact dependence structure of $\mathbf{X}$ itself. When a given dependence structure is chosen, $\rho(S)$ can be computed exactly. However, when the dependence structure is unspecified, $\rho(S)$ can take a range of possible values, which can then be interpreted as a measure of model uncertainty (Cont, 2006) with the absence of information on dependence. The assessment of aggregate risks S with given marginal distributions and partial information on the dependence structure, has been extensively studied in quantitative risk management. A large part of the literature focuses on properties of a specific risk measure when there is no extra information on the dependence structure, for instance: bounds on the distribution function and the Value-at-Risk (VaR) of S were studied by Embrechts et al. (2003), Embrechts and Puccetti (2006) and Wang et al. (2013), among others; convex ordering bounds on S were studied by for example Denuit et al. (1999), Dhaene et al. (2002) and Wang and Wang (2011). Some numerical methods to approximate bounds on risk measures were recently provided by Puccetti and Rüschendorf (2012), Embrechts et al. (2013), and Puccetti (2013). Another direction in the literature has been to study the case when marginals are fixed, and some extra information on the dependence is available; see Cheung and Vanduffel (2013) for convex ordering bounds with given variance; Bernard et al. (2013b) for VaR bounds with a variance constraint; Kaas et al. (2009) for the worst Value-at-Risk with constraints of positively quadrant dependence and some given measures of dependence for $n = 2$; Embrechts and Puccetti (2006) for bounds on the distribution of S when the copula of $\mathbf{X}$ is bounded by a given copula; Tankov (2011) and Bernard et al. (2012), Bernard et al. (2013a) for bounds on S when $n = 2$; see also the Herd index proposed by Dhaene et al. (2012) based on the maximum variance of aggregate risk with estimated marginal variances. We refer to Embrechts and Puccetti (2010) for an overview on risk aggregation with no or partial information on dependence. Note that our work is fundamentally different from the literature on the lower bounds on $\rho(S)$ obtained by conditioning methods (e.g. Rogers and Shi, 1995, Kaas et al., 2000, Valdez et al., 2009).

Another contribution is to show that the convex ordering lower bound gives the explicit expression of the infimum and supremum of VaR (and proves the intuition behind the numerical bounds obtained for example by Embrechts et al., 2013, Puccetti and Rüschendorf, 2012). Convex ordering bounds are also directly related to bounds on convex expectations[1] and on general law-invariant convex risk measures, including coherent risk measures. Convex expectations appear naturally in many practical problems such as basket options (Tankov, 2011; D'Aspremont and El Ghaoui, 2006; Hobson et al., 2005; Albrecher et al., 2008), discrete variance options pricing (Keller-Ressel and Clauss, 2012), stop-loss premiums for aggregate risk, variances and expected utilities. Examples are discussed extensively in Section 5. *Coherent risk measures* were introduced in Artzner et al. (1999) and later extended by Delbaen (2002) and Kusuoka (2001), among others. See also McNeil et al. (2005) for an overview. In Section 5, we will discuss how convex ordering bounds lead to bounds on convex and coherent risk measures. An important application for the financial industry is to obtain bounds on the coherent risk measure Tail-Value-at-Risk[2] (TVaR) of a joint portfolio $S = X_1 + \cdots + X_n$, when the dependence between individual assets $X_1, \ldots, X_n$ is unknown. More details and applications are given in Section 5.

The rest of the paper is organized as follows. In Section 2 we introduce the concept of *admissible risk class* and derive its properties. The main results of this paper focus on this class. Section 3 provides a new convex ordering lower bound over the admissible risk class, for both homogeneous and heterogeneous risks. It is shown that under some conditions, this bound is sharp in the homogeneous case. Section 4 gives a connection between the convex ordering lower bound and bounds on the Value-at-Risk. Bounds on convex risk measures and other applications are then given in Section 5. Some numerical illustrations can be found in Section 6. Concluding remarks are given in Section 7.

2. Admissible risk

Assume that all random variables live in a general atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$. This means that for all $A \subset \Omega$ with $\mathbb{P}(A) > 0$, there exists $B \subsetneq A$ such that $\mathbb{P}(B) > 0$. The atomless assumption is very weak: in our context it is equivalent to that there exists at least one continuously distributed random variable in this space (roughly, $(\Omega, \mathcal{A}, \mathbb{P})$ is not a finite space). In particular, it does not prevent discrete variables to exist. In such a probability space, we can generate sequences of independent random vectors with any distribution. We denote by $L^0(\Omega, \mathcal{A}, \mathbb{P})$ the set of all random variables defined in the atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$. See Proposition 6.9 of Delbaen (2002) for details of atomless probability spaces. More discussions on risk measures defined on such spaces can also be found in this paper.

Throughout the paper, we call *aggregate risk* the sum $S = X_1 + \cdots + X_n$ where X_i are non-negative random variables (individual risks) and n is a positive integer. Here the non-negativity is assumed just for the convenience of our discussion.

As already mentioned before, we consider that for each $i = 1, \ldots, n$ the distribution of X_i is known while the joint distribution of $\mathbf{X} := (X_1, X_2, \ldots, X_n)$ is unknown. In other words, marginal distributions of individual risks X_i are given and their dependence structure (copula) is unspecified. To formulate the problem mathematically, define the *Fréchet class* $\mathfrak{F}_n(F_1, \ldots, F_n)$ as the set of random vectors with given marginal distributions $F_1, \ldots, F_n$,

$$\mathfrak{F}_n(F_1, \ldots, F_n) = \{\mathbf{X} : X_i \sim F_i,\ i = 1, \ldots, n\},$$

where $X_i \sim F_i$ means that $X_i \in L^0(\Omega, \mathcal{A}, \mathbb{P})$ has the distribution F_i. The Fréchet class is the most natural setup to describe the case when marginal distributions are known and the dependence is unspecified. It was used in the literature when studying copulas and dependence in risk management; we refer to recent review

[1] A *convex (concave) expectation* is computed as $\mathbb{E}[f(S)]$ where $f : \mathbb{R} \to \mathbb{R}$ is a convex (concave) function.

[2] The TVaR of S at level $p \in [0, 1)$ is defined as $\text{TVaR}_p(S) = \frac{1}{1-p}\int_p^1 \text{VaR}_\alpha(S)\text{d}\alpha$, where VaR is the Value-at-Risk measure.