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Viscosity solution and impulse control of the diffusion model with reinsurance and fixed transaction costs

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HIGHLIGHTS

- Study an optimal impulse control problem with proportional transaction costs.
- Prove the value function is a unique solution to the associated HJB equation.
- Establish the regularity property of the viscosity solution.
- Derive the closed forms of the value function and the optimal strategy.

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ABSTRACT

We consider an optimal impulse control problem on reinsurance, dividend and reinvestment of an insurance company. To close reality, we add fixed and proportional transaction costs to this problem. The value of the company is associated with expected present value of net dividends pay out minus the net reinvestment capitals until ruin time. We focus on non-cheap proportional reinsurance. We prove that the value function is a unique solution to associated Hamilton–Jacobi–Bellman equation, and establish the regularity property of the viscosity solution under a weak assumption. We solve the non-uniformly elliptic equation associated with the impulse control problem. Finally, we derive the value function and the optimal strategy of the control problem.

1. Introduction

Optimal dividend strategy, as one major public concern to assess the stability of companies that take on risks, has been a long standing problem and has also become an increasingly popular topic in actuarial research. Its origin can be traced as early as the work of Finetti (1957), where a discrete-time model for optimal dividend was introduced. Finetti stated that the optimal strategy is a barrier strategy, and determined the optimal level of the barrier. The work laid the foundation of study of dividend strategies. Since then the optimal strategies related to ruin problems have received renewed interests in the literature. Some early works include Borch (1967, 1969) and Gerber (1972, 1979), and a survey paper Avanzi (2009) and references therein. Some recent works are He and Liang (2008, 2009), Liang and Huang (2011), Alvarez and Lempa (2008), Avanzi and Gerber (2008) and Albrecher and Gerber (2009), Paulsen and Gjessing (1997), Asmussen and Taksar (1997),

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Højgaard and Taksar (2004), Gerber and Shiu (1998, 2003a,b, 2004) and references therein.

On the other hand, reinsurance and reinvestment are also two important approaches for the insurance company to earn profit and reduce risk in the real financial market. The three popular types of reinsurance strategies are stop-loss, proportional reinsurance and excess-of-loss reinsurance. Some literatures on stop-loss and excess-of-loss reinsurance include Asmussen et al. (2000), Choulli et al. (2001), Hürlimann (2006), Mnif and Sulem (2005), Paulsen and Gjessing (1994), Zhang et al. (2007), and Meng and Siu (2011a,b). The works on optimal proportional reinsurance include Asmussen and Taksar (1997), Højgaard and Taksar (2004), Azcue and Muler (2005), Schmidt (2004), Promislow and Young (2005) and Taksar (2000a,b). We refer the readers to Porteus (1977) in a discrete-time modeling framework and Sethi and Taksar (2002) in a continuous-time diffusion model for details on optimal reinvestment problems. Løkka and Zervos (2008) studied the combined optimal dividend and reinvestment problem by taking into account the possibility of bankruptcy. He and Liang (2008, 2009) incorporated the proportional reinsurance strategy in the combined dividend and reinvestment problem using both singular and mixed singular-impulse controls.

The viscosity solution is very useful to characterize the value function when it is not in C^2. For more motivations and background of viscosity solution, we refer the readers to Crandall et al. (1992). The study of viscosity solution to Hamilton–Jacobi–Bellman (abbr. HJB) equation on the value function is now very popular, see Guo and Wu (2009) and Meng and Siu (2011a,b). However, Guo and Wu (2009) assume a strong condition: strict positive lower bounds for the transaction cost functions, while Mnif and Sulem (2005) consider optimal excess-of-loss reinsurance policies in the combined dividend and reinvestment problem, but they do not embody fixed transaction costs.

Motivated by these works, we consider an optimal impulse control problem in both the cheap and non-cheap proportional reinsurance strategies with fixed and proportional transaction costs. In this paper, we provide a rigorous and detailed mathematical analysis for the insurance company on the combined effect of optimal proportional reinsurance, reinvestment and dividend strategies. The value of the company is associated with expected present value of net dividends pay out minus the net reinvestment capitals until ruin time. We prove that the value function is a unique solution to associated HJB equation, and establish the regularity property of the viscosity solution under a weak assumption. However, we have to tackle solvability of a kind of the non-uniformly elliptic equations associated with the impulse control problem in the set of smooth functions when we deal with the regularity property. Unfortunately, Lions (1983) only gave the solvability of this kind of problems under the uniformly elliptic condition in the set of smooth functions, Lions' work does not cover our case, and it is the key point to establish the regularity property. So we will present a rigorous proof on the solvability of the non-uniformly elliptic equations associated with the control problem in the set of smooth functions. Finally, we derive explicit solutions of the value functions and find the optimal strategies of the impulse control problems.

The rest of the paper is organized as follows. In Section 2, we formulate the optimal impulse control problem. In Section 3, we give some properties of the value function. In Section 4, we establish the viscosity solution of HJB equations. In Section 5, we prove the regularity property of the viscosity solution under a weak assumption, and in Section 6 we give the solution to the optimal impulse control problem. Section 7 is a conclusion.

2. Risk model and impulse control problem for insurance company with proportional reinsurance policy

We consider a continuous-time insurance risk model with an infinite-time horizon $T = [0, \infty]$. To give a mathematical foundation of our impulse control problem, as usual we fixed a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mathbf{P})$, and $\{\mathrm{B}_t,\ t \in T\}$ is a standard Brownian motion on this probability space, where $\{\mathcal{F}_t\}_{t\in T}$ is an filtration satisfying the usual conditions, $\mathcal{F}_t$ represents the information available at time t and any decision made up t is based on this information. For the sake of completeness and the intuition of motivation, we start from the classical Cramér–Lundberg model. We denote $\{\mathrm{R}_t,\ t \in T\}$ as the reserve of the insurance company without dividend and reinsurance policies and the reserve process is given by $\mathrm{R}_t = \mathrm{R}_0 + pt - \sum_{i=1}^{\mathrm{N}_t} \mathrm{U}_i$, where R_0 is the initial capital of the company, N_t is a Poisson process with intensity β representing the number of the claims occurring up to t, U_i is the size of the ith claim, and we assume that $\mathrm{U}_i,\ i \geq 1$, are i.i.d. random variables with common distribution F having finite first moment μ and second moment σ^2. We also assume that $\{\mathrm{N}_t\}$ and $\{\mathrm{U}_i\}$ are independent under $\mathbf{P}$. So the premium rate is determined by expected value principle, i.e.,

$$p = (1 + \eta)\beta\mu,$$

where η is relative safety loading and $\eta > 0$.

The insurance company can transfer a portion of the risk attributed to the insurance claim U_i by entering a reinsurance contract. Let a be the retention level and let U_i^a denote the portion of the claims held by the insurer. We assume the reinsurance company uses a safety loading proportion to η with proportional factor $\nu(a)$ depending only on a. Then the reserve process of the modified Cramér–Lundberg model with reinsurance is described by

$$\mathrm{R}_t^{(a,\eta)} = \mathrm{R}_0 + p^{(a,\eta)}t - \sum_{i=1}^{\mathrm{N}_t} \mathrm{U}_i^a,$$

where the premium rate is

$$\begin{aligned} p^{(a,\eta)} &= (1+\eta)\beta\mu - (1+\nu(a)\eta)\beta[\mu - \mathbf{E}\mathrm{U}_i^a] \\ &= \beta\mathbf{E}\mathrm{U}_i^a + \eta\beta(\mu - \nu(a)[\mu - \mathbf{E}\mathrm{U}_i^a]). \end{aligned}$$

Then it is easy to see that (cf. Liang and Sun, 2011)

$$\{\eta \mathrm{R}^{(a,\eta)}_{t/\eta^2}\}_{t\geq 0} \xrightarrow{\mathfrak{D}} BM(\mu(a), \sigma^2(a))$$

in $D[0, \infty]$ (the space of right continuous functions with left limits endowed with the Skorohod topology) as $\eta \downarrow 0$, where

$$\mu(a) = \beta(\mu - \nu(a)[\mu - \mathbf{E}\mathrm{U}_i^a]), \qquad \sigma^2(a) = \beta\mathbf{E}\mathrm{U}_i^{(a)^2}.$$

So the limiting model is suitable to describe big portfolios, we refer the reader to Grandell (1990) for details of this motivation.

If $\nu(a) \equiv 1$, it means cheap reinsurance one, and we mean non-cheap reinsurance one if $\nu(a) > 1$. For the proportional reinsurance, $\mathrm{U}_i^a = a\mathrm{U}_i$. Therefore we have

$$\mu(a) = \beta(\mu - (1-a)\nu(a)\mu), \qquad \sigma^2(a) = \beta a^2\sigma^2.$$

Without loss of generality, we assume the $\beta = 1$ and $\nu(a)$ is a constant. Thus the diffusion process approximating the Cramér–Lundberg model is governed by

$$dr(t) = (\mu - \lambda(1-a))dt + a\sigma dB_t,$$

where $\lambda = \nu(a)\mu$, and so $\lambda \geq \mu$.

In our model, we consider the retention level to be the control parameter selected at each time t by the insurance company.