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Second-order tail asymptotics of deflated risks

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HIGHLIGHTS

- Second-order tail asymptotics of deflated risks are investigated for three MDA cases.
- Several examples illustrate the increased accuracy achieved using our results.
- We apply our results to VaR, tail probability and aggregated risk.
- Second-order properties of common insurance risks are derived.

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ABSTRACT

Random deflation of risk models is an interesting topic for both theoretical and practical actuarial problems. In this paper, we investigate second-order tail asymptotics of the deflated risk $X = RS$ under the assumptions of second-order regular variation on the survival functions of the risk R and the deflator S. Our findings are applied to derive second-order expansions of Value-at-Risk. Further we investigate the estimation of small tail probability for deflated risks and then discuss the asymptotics of the aggregated deflated risk.

1. Introduction

Let R be a non-negative random variable (rv) with distribution function (df) F being independent of the rv $S \in (0, 1)$ with df G. If R models the loss amount of a financial risk, and S models a random deflator for a particular time-period, then the product $X = RS$ represents the deflated value of R at the end of the time-period under consideration. Random deflation is a natural phenomena in various actuarial applications attributed to the time-value of money. When large values or extremes are of interest, for instance for reinsurance pricing and risk management purposes, it is important to link the behaviors of the risk R and the random deflator S. Intuitively, we expect that large values observed for R are not significantly influenced by the random deflation. However, this is not always the case; a precise analysis driven by some extreme value theory models is given in Tang and Tsitsiashvili (2004), Tang (2006, 2008), Hashorva et al. (2010), Arendarczyk and Dębicki (2011), Tang and Yang (2012), Zhu and Li (2012), Hashorva (2013), Yang and Hashorva (2013), Yang and Wang (2013), and the references therein. The results of the aforementioned papers are obtained mainly under a first-order asymptotic condition for the survival function or the quantile function in extreme value theory, i.e., the df F under consideration belongs to the max-domain of attraction (MDA) of a univariate extreme value distribution Q_γ, $\gamma \in \mathbb{R}$, abbreviated as $F \in D(Q_\gamma)$, which means that

$$F^n(a_n x + b_n) \to \exp\left(-(1+\gamma x)^{-1/\gamma}\right) =: Q_\gamma(x),$$
$$1 + \gamma x > 0, n \to \infty \quad (1.1)$$

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holds for some constants $a_n > 0$ and $b_n \in \mathbb{R}$, $n \geq 1$, see Resnick (1987). The parameter γ is called the extreme value index; according to $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$, the df F belongs to the MDA of the Fréchet distribution, the Gumbel distribution and the Weibull distribution, respectively.

In order to derive some more informative asymptotic results, second-order regular variation (2RV) conditions are widely used in extreme value theory. Here we only mention de Haan and Resnick (1996) for the uniform convergence rate of F^n to its ultimate extreme value distribution Q_γ under 2RV, and Beirlant et al. (2009, 2011), Ling et al. (2012) and the references therein for the asymptotic distributions of the extreme value index estimators under consideration.

Indeed, almost all the common loss distributions including log-gamma, absolute t, log-normal, Weibull, Benktander II, and Beta (cf. Table 2 in the Appendix) possess 2RV properties; actuarial applications based on those properties are developed in the recent contributions Hua and Joe (2011), Mao and Hu (2012, 2013) and Yang (2014).

The main contributions of this paper concern the second-order expansions of the tail probability of the deflated risk $X = RS$ which are then illustrated by several examples. Our main findings are utilized for the formulations of three applications, namely approximation of Value-at-Risk, estimation of small tail probability of the deflated risk, and the derivation of the tail asymptotics of aggregated risk under deflation.

The rest of this paper is organized as follows. In Section 2 we present our main results under second-order regular variation conditions. Section 3 shows the efficiency of our second-order asymptotics through some illustrating examples. Section 4 is dedicated to three applications. The proofs of all results are relegated to Section 5. We conclude the paper with a short Appendix.

2. Main results

We start with the definitions and some properties of regular variation followed by our principal findings. A measurable function $f : [0, \infty) \to \mathbb{R}$ with constant sign near infinity is said to be of second-order regular variation with parameters $\alpha \in \mathbb{R}$ and $\rho \leq 0$, denoted by $f \in 2\mathrm{RV}_{\alpha,\rho}$, if there exists some function A with constant sign near infinity satisfying $\lim_{t\to\infty} A(t) = 0$ such that for all $x > 0$ (cf. Bingham et al. (1987) and Resnick (2007))

$$\lim_{t\to\infty} \frac{f(tx)/f(t) - x^\alpha}{A(t)} = x^\alpha \int_1^x u^{\rho-1}\, du =: H_{\alpha,\rho}(x). \tag{2.1}$$

Here, A is referred to as the auxiliary function of f. Note that (2.1) implies $\lim_{t\to\infty} f(tx)/f(t) = x^\alpha$, i.e., f is regularly varying at infinity with index $\alpha \in \mathbb{R}$, denoted by $f \in \mathrm{RV}_\alpha$. RV_0 is the class of slowly varying functions. For f eventually positive, it is of second-order Π-variation with the second-order parameter $\rho \leq 0$, denoted by $f \in 2\mathrm{ERV}_{0,\rho}$, if there exist some functions a and A with constant signs near infinity and $\lim_{t\to\infty} A(t) = 0$ such that for all x positive

$$\lim_{t\to\infty} \frac{\frac{f(tx)-f(t)}{a(t)} - \ln x}{A(t)} = \psi(x) := \begin{cases} \dfrac{x^\rho - 1}{\rho}, & \rho < 0, \\ \dfrac{\ln^2 x}{2}, & \rho = 0. \end{cases} \tag{2.2}$$

(cf. Resnick (2007)), where the functions a and A are referred to as the first-order and the second-order auxiliary function of f, respectively. From Theorem B.3.1 in de Haan and Ferreira (2006) we see that $a \in 2\mathrm{RV}_{0,\rho}$ with auxiliary function A, and that $|A| \in \mathrm{RV}_\rho$. In fact, (2.2) implies $\lim_{t\to\infty}(f(tx) - f(t))/a(t) = \ln x$ for all $x > 0$, which means f is Π-varying with auxiliary function a, denoted by $f \in \Pi(a)$.

We shall keep the notation of the Introduction for R and $S \in (0, 1)$, denoting their df's by F and G, respectively, whereas the df of $X = RS$ will be denoted by H. Throughout this paper, let $\bar{F}_0 = 1 - F_0$ denote the survival function of a given df F_0.

Next, we present our main results. Theorem 2.1 gives a second-order counterpart of Breiman's Lemma (see Breiman (1965)) while Theorems 2.3 and 2.6 include refinements of the tail asymptotics of products derived in Hashorva et al. (2010).

Theorem 2.1. *If $F \in D(Q_{1/\alpha_1})$ satisfies $\bar{F} \in 2\mathrm{RV}_{-\alpha_1,\tau_1}$ with auxiliary function $\tilde{A}$ for some $\alpha_1 > 0$ and $\tau_1 \leq 0$, then*

$$\frac{\bar{H}(x)}{\bar{F}(x)} = \mathbb{E}\{S^{\alpha_1}\}[1 + \mathcal{E}(x)], \tag{2.3}$$

where $\mathcal{E}(x) = \left(\mathbb{E}\left\{S^{\alpha_1-\tau_1}\right\}/\mathbb{E}\{S^{\alpha_1}\} - 1\right)\tilde{A}(x)/\tau_1(1 + o(1))$ *as $x \to \infty$, and thus $\bar{H} \in 2\mathrm{RV}_{-\alpha_1,\tau_1}$ with auxiliary function*

$$A^*(x) = \frac{\mathbb{E}\left\{S^{\alpha_1-\tau_1}\right\}}{\mathbb{E}\{S^{\alpha_1}\}}\tilde{A}(x).$$

Remark 2.2. (a) The expression for $\tau_1 = 0$ is understood throughout this paper as its limit as $\tau_1 \to 0$.

(b) Under the assumptions of Theorem 2.1, Breiman's Lemma only implies

$$\frac{\bar{H}(x)}{\bar{F}(x)} = \mathbb{E}\{S^{\alpha_1}\}[1 + \mathcal{E}^*(x)]$$

with $\lim_{x\to\infty} \mathcal{E}^*(x) = 0$, while the error term $\mathcal{E}(x)$ in (2.3) not only converges to 0 as $x \to \infty$, but it shows also the speed of convergence being determined by $\tilde{A}(x)$.

Next, we shall consider the cases that F belongs to the MDA of the Gumbel distribution and the Weibull distribution, respectively. Compared to the heavy-tail case above, we need to impose some assumptions on the tail of S; see Hashorva et al. (2010). In our setting, we strengthen L (see (2.4) below for an accurate definition) to be of second-order regular variation.

We shall write $Y \sim Q$ for some rv Y with df Q, whereas $Q^{\leftarrow}$ denotes the generalized left-continuous inverse of Q (also for Q which are not dfs). Since both H and F have the same upper endpoint $x_H = x_F := \sup\{y : F(y) < 1\}$, then all the limit relations below are for $x \uparrow x_F$ unless otherwise specified. Further, for some $\alpha_2 > 0$ we set

$$L(x) = x^{\alpha_2}\bar{G}\left(1 - \frac{1}{x}\right),$$
$$K(\alpha_2, \rho) = \begin{cases} \dfrac{(1-\rho)^{-\alpha_2} - 1}{\rho}\Gamma(\alpha_2 + 1), & \rho < 0, \\ \dfrac{\alpha_2\Gamma(\alpha_2 + 2)}{2}, & \rho = 0, \end{cases} \tag{2.4}$$

where $\Gamma(\cdot)$ is the Euler Gamma function, and define

$$w(x) = \frac{1}{\mathbb{E}\{R - x|R > x\}}, \qquad \eta(x) = xw(x). \tag{2.5}$$

Hereafter the generalized left-continuous inverses of $1/\bar{F}$ and $1/\bar{H}$ are denoted respectively by

$$U = U_R = (1/\bar{F})^{\leftarrow} \quad \text{and} \quad U_X = (1/\bar{H})^{\leftarrow}.$$

Theorem 2.3. *Let F be strictly increasing and continuous in the left neighborhood of x_F and let $U \in 2\mathrm{ERV}_{0,\rho}$, $\rho \leq 0$ with auxiliary functions $1/w(U)$ and $\tilde{A}$. If $L \in 2\mathrm{RV}_{0,\tau_2}$, $\tau_2 < 0$ with auxiliary function A, then*

$$\frac{\bar{H}(x)}{\bar{F}(x)\bar{G}(1 - 1/\eta(x))} = \Gamma(\alpha_2 + 1) + \mathcal{E}(x), \tag{2.6}$$