Insurance: Mathematics and Economics 53 (2013) 48–63

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Actuarial applications of the linear hazard transform in mortality immunization

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HIGHLIGHTS

- Magnitude-free mortality durations and convexities are defined and derived.
- The weights of a portfolio are determined by duration/convexity matching strategies.
- The matching strategies can be used for hedging mortality/longevity risks.
- A portfolio of two products of life insurance and annuity is always feasible.
- Three-product portfolios are not always feasible under some conditions.

ARTICLE INFO

Article history:
Received October 2011
Received in revised form
February 2013
Accepted 26 March 2013

Keywords:
Proportional hazard transform
Linear hazard transform
Immunization
Hedging
Duration
Convexity

ABSTRACT

In this paper, we apply the linear hazard transform to mortality immunization. When there is a change in mortality rates, the respective surplus (negative reserve) changes for life insurance and annuity policies lead to oppositive sign changes, which provides mortality hedging strategies with a portfolio of life insurance and annuity policies. We first show that by the strategy of matching duration of the weighted surplus at time 0, the surplus changes at time 0 for both portfolios P^{TP} (the n-year term life insurance and the n-year pure endowment) and P^{WA} (the n-payment whole life insurance and the n-year deferred whole life annuity) in response to a proportional or parallel shift in the underlying force of mortality are always negative. Next, we prove that the term life insurance, the whole life insurance and the deferred whole life annuity cannot always form a feasible portfolio (feasibility means that all the weights of the product members of a portfolio are positive) by the strategy of matching two durations or one duration and one convexity of the weighted surplus at time 0. Finally, numerical examples including figures and tables are exhibited for illustrations.

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1. Introduction

In actuarial science, the Gompertz, Makeham and Weibull laws are common forces of mortality for modeling human mortality rates. The force of mortality $\mu_x(\cdot)$ for an insured aged x plays an important role in actuarial pricing since ${}_kp_x$, the probability that the insured survives k years, can be expressed in terms of $\mu_x(\cdot)$, that is, ${}_kp_x = e^{-\int_0^t \mu_x(s)ds}$. Actually, $\mu_x(\cdot)$ is a hazard rate because $\mu_x(t) = -[\partial_t p_x/\partial t]/{}_tp_x$ and ${}_tp_x$ is a survival distribution. The proportional transform of $\mu_x(\cdot)$ is obtained by multiplying it with a constant $(1+\alpha)$ to form $(1+\alpha)\mu_x(t)$. Its actuarial applications can be widely found in the literature in 1995–1999. For example, Wang (1995) gave a premium calculation principle based on the proportional hazard transform for insurance pricing and increased limits ratemaking. It is called Wang's premium calculation principle which has many properties for measuring risks. Wang (1996) showed that his premium principle resembles the risk-neutral valuation in financial economics, but differs from the traditional utility theory approach.

When a constant β is further added to the proportional hazard transform, the linear hazard transform is formed as $(1+\alpha)\mu_x(t) + \beta$. Actuarial applications of the linear hazard transform can be found in Tsai and Jiang (2011, 2013). The former combined the assumption of α-power approximation with the linear hazard transform to approximate the net single premium of a continuous life insurance policy in terms of the net single premiums of discrete ones. Moreover, Macaulay duration, modified duration and dollar duration, all measuring the sensitivity of the price of a life insurance or annuity policy to movements of the force of mortality under the linear hazard transform, were defined and investigated. The

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http://dx.doi.org/10.1016/j.insmatheco.2013.03.013

latter applied the linear hazard transform to mortality fitting and prediction by assuming that there is a linear relation plus an error term between the forces of mortality for two mortality sequences. Then the parameters α and β of the linear relation are obtained by fitting the target sequence by the source one with the method of regression. When these two mortality sequences are for two different years, the parameters of the linear hazard transform can be used to make a sequence of forward or backward mortality prediction for the year we are interested in.

In finance, interest rate immunization ensures that the value of a portfolio will not be affected in response to a change in interest rates. Consider a block of life insurance policies and associated assets. Let $A_t \geq 0$ and $L_t \geq 0$ be the asset and liability cash flows expected at time t, respectively, and $N_t = A_t - L_t$ be the net cash flow at time t. If the force of interest is δ, then the present value of the net cash flows at time 0 is equal to $S_0(\delta) = \sum_{t\geq 0} N_t \cdot e^{-\delta t}$. One of interest rate immunization problems is to study what the conditions are on the cash flows such that

$$S_0(\delta+\epsilon) \geq S_0(\delta) \tag{1.1}$$

when the force of interest δ is changed to $\delta+\epsilon$ where ϵ is a positive or negative number.

By expanding $S_0(\delta+\epsilon)$ to $S_0(\delta+\epsilon) = S_0(\delta) + \epsilon \cdot S_0'(\delta) + \epsilon^2 \cdot S_0''(\delta+\zeta)/2$ where ζ is some value between 0 and ϵ, Redington (1952) demonstrated that

$$S_0'(\delta) = -\sum_{t\geq 0} t \cdot N_t \cdot e^{-\delta t} = 0$$

and

$$S_0''(\eta) = \sum_{t\geq 0} t^2 \cdot N_t \cdot e^{-\eta t} \geq 0$$

for all η imply (1.1). Note that $-S_0'(\delta)$ and $S_0''(\delta)/S_0(\delta)$ are so-called dollar duration and convexity of $S_0(\delta)$ in finance, respectively. Fisher and Weil (1971) studied this immunization problem for single liability L_T paid at time T and relaxed Redington's assumption of constant force of interest. The Fisher–Weil immunization theorem states that if the assets and the liabilities have the same present values and durations, that is,

$$\sum_{t\geq 0} A_t \cdot e^{-\int_0^t \delta(s)ds} = L_T \cdot e^{-\int_0^T \delta(s)ds},$$

and

$$\sum_{t\geq 0} t \cdot A_t \cdot e^{-\int_0^t \delta(s)ds} = T \cdot L_T \cdot e^{-\int_0^T \delta(s)ds},$$

then

$$\sum_{t\geq 0} A_t \cdot e^{-\int_0^t (\delta(s)+\epsilon)ds} \geq L_T \cdot e^{-\int_0^T (\delta(s)+\epsilon)ds}. \tag{1.2}$$

Shiu (1987) extended the Fisher–Weil immunization theorem by assuming that the shift ϵ is a function of time as well. He showed that if $[\epsilon(t)]^2 \geq \epsilon'(t)$ for each t then (1.2) holds, and if $[\epsilon(t)]^2 \leq \epsilon'(t)$ for each t, then the inequality in (1.2) is reversed. Shiu (1988) studied the multiple-liability immunization problem and demonstrated that the separate immunization of each liability outflow is not only a sufficient condition but also a necessary one for the immunization of multiple liabilities. Shiu (1990) proposed $\sum_{t\geq 0} t \cdot n_t = 0$ and $\sum_{t\geq 0}[(t-w)_+] \cdot n_t \geq (\leq)\, 0$ for all positive w where $n_t = N_t \cdot e^{-\int_0^t \delta(s)ds}$ as two sufficient conditions for

$$\begin{aligned} S_0(\delta+\epsilon) &= \sum_{t\geq 0} N_t \cdot e^{-\int_0^t [\delta(s)+\epsilon(s)]ds} \geq (\leq) \sum_{t\geq 0} N_t \cdot e^{-\int_0^t \delta(s)ds} \\ &= S_0(\delta) \end{aligned} \tag{1.3}$$

provided that the shift function $\epsilon(\cdot)$ is constant as in Redington's model. Condition $\sum_{t\geq 0}[(t-w)_+] \cdot n_t \geq (\leq)\, 0$ is difficult to verify since it needs to be satisfied for each positive w. Instead, since $\sum_{t\geq 0} n_t = 0$ and $\sum_{t\geq 0} t \cdot n_t = 0$ imply that sequence $\{n_t : t \geq 0\}$ has at least two sign changes, Shiu (1990) applied Lemma 4 (p. 202) of Goovaerts et al. (1984) and showed that if sequence $\{n_t : t \geq 0\}$ has exactly two sign changes and the pattern is of the form $+, -, +$ $(-, +, -)$ then $\sum_{t\geq 0} \phi(t)\, n_t \geq (\leq)\, 0$ for all convex functions $\phi(\cdot)$ including $\phi(t) = (t-w)_+$ for each positive w.

Mortality immunization ensures that the value of surplus (negative reserve) of an insurance portfolio will not be negatively affected when a change in mortality rates is made. The underlying portfolio consists of life insurance and annuity policies since their surpluses are affected reversely in response to a change in mortality rates. Natural hedging for mortality risks uses this characteristic of life insurance and annuities to a change in mortality rates to hedge against unexpected changes in future benefits. Cox and Lin (2007) found empirical evidence that annuity insurers with the natural hedging strategy charge smaller premiums than otherwise similar insurers. Wang et al. (2010) combined an immunization approach with a stochastic mortality model to compute an optimal life insurance–annuity product mix ratio to hedge against longevity risks. Inspired by Shiu (1990), we apply the linear hazard transform to study some similar problems focusing on mortality immunization. The verifiable sufficient conditions above for (1.1) proposed by Shiu (1990) are still based on a constant shift ϵ as in Redington's model; otherwise the sufficient conditions associated with a general shift function of time, $\epsilon(t)$, might be difficult to verify. First, consider the force of mortality $\mu_x(t)$ for the insured aged x. The linear transform of $\mu_x(t)$, $(1+\alpha) \cdot \mu_x(t) + \beta$, can be interpreted as that the force of mortality $\mu_x(t)$ is shifted proportionally and constantly to $(1+\alpha) \cdot \mu_x(t) + \beta$. In this case, the shift function $\epsilon(t) = \alpha \cdot \mu_x(t) + \beta$ is not constant unless $\alpha = 0$. Then we are going to propose some sufficient conditions such that the sign of the surplus change due to a change in α or β is always positive or negative.

The remainder of the paper proceeds as follows. In Section 2, we define durations and convexities of the net single premium and the surplus of a life insurance/annuity product at time zero with respect to a proportional or parallel shift in the underlying force of mortality. We prove that the signs of both durations and convexities of the net single premium and the surplus at time zero for of life insurance and annuity products are opposite. Section 3 studies mortality immunization for two portfolios, each of which consists of an annuity and a life insurance products. When the duration matching strategy is adopted, we show that the change of the weighted surplus at time zero for each of these two portfolios in response to a proportional or parallel change in the underlying force of mortality is always negative. In Section 4, we investigate mortality immunization for a portfolio comprising three products — the term life insurance and the whole life insurance plus the whole life annuity. When the strategy of matching two durations or one duration and one convexity is adopted, we demonstrate that not all of the three weights for the portfolio are positive, that is, the portfolio is not always feasible. Finally, relevant numerical examples are given for illustration in Sections 3 and 4.

2. Duration and convexity

Duration is a corner stone of the strategy for immunization. Macaulay duration, modified duration and dollar duration are three common types of durations in finance, which measure the sensitivity of the price of an asset to a parallel shift in the interest rate. Durations are widely used in asset and liability management to help match liabilities with assets in order to stabilize cash flows in the future. The duration matching strategy is a common approach of hedging interest rate risks for an interest-sensitive

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