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Optimal time-consistent investment and reinsurance strategies for mean–variance insurers with state dependent risk aversion[☆]

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HIGHLIGHTS

- We study an investment and reinsurance problem within a game theoretic framework.
- The risk aversion depends dynamically on current wealth instead of a constant one.
- We provide an analytical solution for the time-consistent strategies.

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ABSTRACT

In this paper, we study an insurer's optimal time-consistent strategies under the mean–variance criterion with state dependent risk aversion. It is assumed that the surplus process is approximated by a diffusion process. The insurer can purchase proportional reinsurance and invest in a financial market which consists of one risk-free asset and multiple risky assets whose price processes follow geometric Brownian motions. Under these, we consider two optimization problems, an investment–reinsurance problem and an investment-only problem. In particular, when the risk aversion depends dynamically on current wealth, the model is more realistic. Using the approach developed by Björk and Murgoci (2009), the optimal time-consistent strategies for the two problems are derived by means of corresponding extension of the Hamilton–Jacobi–Bellman equation. The optimal time-consistent strategies are dependent on current wealth, this case thus is more reasonable than the one with constant risk aversion.

1. Introduction

In order to control and manage risk, reinsurance and investment are effective ways for insurers. In the past decades, optimal reinsurance and investment problems for insurers have attracted much attention in the actuarial literature. Specifically, the insurers usually consider the control of purchasing proportional reinsurance to reduce risk exposure for optimal reinsurance problems. As investment is an increasingly important element in insurance business, the insurers also consider the control of investing in the financial market for optimal investment problems. Further, considering both the reinsurance and investment has become more and more popular in recent years.

Among these, the main approaches are stochastic control theory and related methodologies. See, for example, Browne (1995) considers a model in which the surplus process is modeled by a Brownian motion with drift, the price process of the risky asset is described by a geometric Brownian motion, and the optimal investment strategy for minimizing the probability of ruin was obtained. Yang and Zhang (2005) study the same optimal investment problem with jump–diffusion risk process for an insurer who maximizes the expected exponential utility of the terminal wealth or maximizes the survival probability. For other related works we refer the reader to Taksar and Markussen (2003),

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Xu et al. (2008), Cao and Wan (2009), Chen et al. (2010), Bai and Guo (2008) and Gu et al. (2010) and references therein.

Under the mean–variance criterion, the optimal investment and reinsurance problems for insurers were also considered by many authors. It is well known that the mean–variance approach proposed by Markowitz (1952) is viewed as the foundation of modern finance theory and inspired literally hundreds of extensions and applications. Among others, Li and Ng (2000) developed an embedding technique to change the originally time-inconsistent mean–variance problem into a stochastic LQ control problem in a discrete-time setting. And Zhou and Li (2000) extend this technique to the continuous-time case by applying an indefinite stochastic linear–quadratic control approach. Moreover, under the mean–variance criterion, optimal portfolio selection for insurers has attracted an increasing interest and attention in recent years. For example, Bäuerle (2005) considers the optimal proportional reinsurance problem under the mean–variance criterion where he assumes that the surplus process of an insurer is described by Craḿer–Lundberg (C–L) model, and solves this problem by adopting the stochastic control approach. On this topic, we can see Delong and Gerrard (2007), Bai and Zhang (2008), Zeng et al. (2011), Zeng and Li (2012) and so on.

As we known, the optimal investment and reinsurance problems under the mean–variance criterion in a multi-period or continuous time framework are time inconsistent in the sense that Bellman Optimality Principle does not hold. Time inconsistency is due to the failure of the iterated-expectations property for mean–variance objectives. Hence, dynamic programming approach cannot be easily applied, and even we do not at all be clear what means when one mentions the "optimal". In all these works mentioned above, the authors only studied the pre-committed problem, the pre-committed means that if the decision-makers at time $t = 0$ can commit themselves, they can choose the strategy that is optimal at the initial time, and constrain themselves to abide by it, although they do not see it as optimal in the future.

Another way of handling time inconsistency is to study the problem within a game theoretic framework. The concept of time-inconsistency was first treated formally by Strotz (1955), in which dynamic inconsistent behavior was first formalized analytically. There is similar literature, e.g., Peleg and Menahem (1973) discuss the consumer choice with time-inconsistent preferences. Further work is provided in Pollak (1968), Goldman (1980), Harris and Laibson (2001), Krusell and Smith (2003), Wang and Forsyth (2011) and Kryger and Steffensen (2010).

Recently the time inconsistent problem has again received a lot of attention. For optimal consumption and investment problems with non-exponential discounting have recently been studied from the game theoretic point of view by Ekeland and Lazrak (2006), Ekeland and Pirvu (2007) and Ekeland et al. (2012). Basak and Chabakauri (2010) study the dynamic mean–variance asset allocation and obtain time-consistent strategy with the game theoretic approach. A "general theory of Markovian time inconsistent stochastic control problems" for various forms of time inconsistent objective functions in a Markovian setting is developed by Björk and Murgoci (2009), within this framework the authors derive an extension of the standard dynamic programming equation in the form of a system of equations. Particularly, Björk et al. (2012) precisely study the mean–variance problems with a state dependent risk aversion, in order to have a more realistic model they assume that the risk aversion depends dynamically on current wealth, and the results show that the optimal solution is indeed economically reasonable.

To our knowledge, there is little work in the literature on the time-consistent strategies for the optimal investment and reinsurance problems under the mean–variance criterion. Only Zeng and Li (2011) consider the optimal time-consistent investment and reinsurance strategies for insurers under the mean–variance criterion with the Black–Scholes model, Li et al. (2012) consider the optimal time-consistent investment and reinsurance strategies for an insurer under Heston's stochastic volatility model, and Zeng et al. (2013) study an optimal investment and reinsurance problem incorporating jumps for mean–variance insurers.

However, they consider the problems with constant risk aversion in all those papers. This assumption of a constant risk aversion parameter leads to an equilibrium control. Particularly, the dollar amount invested in the risky asset is independent of current wealth, and we argue that this result is unrealistic from an economic point of view. A person's risk preference certainly depends on his wealth; the obvious intuition is that risk preference should decrease with increasing wealth, so a person with initial wealth of 100 dollars should have a much higher risk aversion value than a one with initial wealth of 100,000,000 dollars. In fact, the relationship between risk aversion and wealth has long been in the research agenda of empirical finance and economics. In order to have a more realistic model we study the case when the risk aversion parameter depends dynamically on current wealth. Of course, the problem under this setting is more challenging and difficult.

In this paper, in order to be economically reasonable, we study an insurer's optimal time-consistent strategies under the mean–variance criterion with state dependent risk aversion. Specifically, we consider an investment–reinsurance problem and an investment-only problem respectively. (In fact, the investment-only problem can been regarded as a special case of the investment–reinsurance problem.) In both problems, the surplus process of the insurer is modeled by a diffusion approximation model, and the financial market consists of one risk-free asset and multiple risky assets whose price processes are followed by geometric Brownian motions. Using the approach developed in Björk and Murgoci (2009), the optimal time-consistent strategies for the two problems are obtained by means of corresponding extension of the Hamilton–Jacobi–Bellman equation. This is the main result of the paper, the result show that the optimal time-consistent investment and reinsurance strategies are dependent on the current wealth. In our opinion, this case is more reasonable than the one with constant risk aversion.

The rest of the paper is organized as follows. In Section 2, we describe the model and some necessary assumptions. Section 3 formulates the problems within a game theoretic framework. In Section 4, we study an investment–reinsurance problem and an investment-only problem with state dependent risk aversion, and derive explicit solution for the two problems. In Section 5, we give a special case of our model, the results in this special case are same as those of Björk et al. (2012). In Section 6, we compare our results with the those of Zeng and Li (2011) for the case with constant risk aversion. In Section 7, we present the numerical results and graphs for illustrative our results. Finally, Section 8 concludes this paper and Appendix is devoted to the proofs of some results in this paper.

2. The model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions, i.e., $(\mathcal{F}_t)_{0\leq t\leq T}$ is right-continuous and $\mathbb{P}$-complete, where T is a positive finite constant representing the time horizon. Suppose that all stochastic processes and random variables are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. In addition, we assume that there are no transaction costs or taxes in the financial market or the insurance market, and trading takes place continuously.