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Long-term behavior of stochastic interest rate models with jumps and memory

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HIGHLIGHTS

- We show the nonnegative property of solutions for a class of stochastic equations.
- We investigate the long-term return for stochastic interest rate models.
- An application to a two-factor CIR model is presented.

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ABSTRACT

The long-term interest rates, for example, determine when homeowners refinance their mortgages in mortgage pricing, play a dominant role in life insurance, decide when one should exchange a long bond to a short bond in pricing an option. In this paper, for a one-factor model, we reveal that the long-term return $t^{-\mu}\int_0^t X(s)\mathrm{d}s$ for some $\mu \geq 1$, in which $X(t)$ follows an extension of the Cox–Ingersoll–Ross model with jumps and memory, converges almost surely to a reversion level which is random itself. Such a convergence can be applied in the determination of models of participation in the benefit or of saving products with a guaranteed minimum return. As an immediate application of the result obtained for the one-factor model, for a class of two-factor model, we also investigate the almost sure convergence of the long-term return $t^{-\mu}\int_0^t Y(s)\mathrm{d}s$ for some $\mu \geq 1$, where $Y(t)$ follows an extended Cox–Ingersoll–Ross model with stochastic reversion level $-X(t)/(2\beta)$ in which $X(t)$ follows an extension of the square root process. This result can be applied to, e.g., how the percentage of interest should be determined when insurance companies promise a certain fixed percentage of interest on their insurance products such as bonds, life-insurance and so on.

1. Introduction

Cox et al. (1985) proposed the short-term interest rate dynamics as

$$\mathrm{d}S(t) = \kappa(\gamma - S(t))\mathrm{d}t + \sigma\sqrt{S(t)}\mathrm{d}W(t)$$

for positive constants κ, γ and σ and standard Brownian motion $\{W(t) : t \geq 0\}$. This model is known as the Cox–Ingersoll–Ross (CIR) model and has some empirically relevant properties, e.g., the randomly moving interest rate is elastically pulled towards the long-term constant value γ. In order to better capture the properties of empirical data, Chan et al. (1992) nested a wide range of models of the short-term interest rate in the framework

$$\mathrm{d}S(t) = \kappa(\gamma - S(t))\mathrm{d}t + \sigma S(t)^{\alpha}\mathrm{d}W(t) \qquad (1)$$

for $\alpha \geq 1/2$ with appropriate restrictions on the parameters κ, γ, α. Here γ, toward which rates drift, stands for the long-term mean of the process, κ means the speed of the drift, σ measures the volatility, and 2α denotes the variance elasticity. In particular, by the $\mathcal{X}^2$ test to the one-month US. Treasury bill yields, Chan et al. (1992) compared the ability of each model with different α to capture the volatility of the term structure, found that the value of α is the most important feature differentiating interest rate models, and revealed that the most successful models in capturing the dynamics of the short-term interest rate are those that allow the volatility of interest rate changes to be highly sensitive to the level of the riskless rate (e.g., $\alpha \geq 1$). Note that the long-term mean γ, the speed of drift κ and the volatility σ are not constants either and there is strong evidence to indicate that they are Markov jump pro-

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cesses. Another generalization of the CIR model is to use regime-switching such as in Ang and Bekaert (2002) and Gray (1996), to name a few. On the other hand, from the economic point of view, there is some evidence indicating that certain events happening before the trading periods influence the current and future asset price, and therefore many scholars introduce delays to the financial models. For example, Arriojas et al. (2007) took delay into consideration for the price process of underlying assets and developed a Black–Scholes type formula. Benhabib (2004) considered a linear, flexible price model, where nominal interest rates are measured by a flexible distributed delay. Stoica (2004) computed the logarithmic utility of an insider when the financial market is modeled by a stochastic delay equation and offered an alternative to the anticipating delayed Black–Scholes formula. However, the mean-reverting square root process cannot explain some empirical phenomena, such as stochastic volatility. To explain these phenomena, jump processes are also used in the financial models, e.g., Bardhan and Chao (1993), Chan (1999), Henderson and Hobson (2003) and Merculio and Runggaldier (1993).

There is extensive literature on quantitative and qualitative properties of the generalized CIR-type models. For instance, different convergence results and the corresponding applications of the long-term returns are found in Deelstra and Delbaen (1995, 1997) and Zhao (2009). Strong convergence of the Monte Carlo simulations are studied in Deelstra and Delbaen (1998) and Wu et al. (2008, 2009), and the representations of solutions are presented in Arriojas et al. (2007) and Stoica (2004). Deelstra and Delbaen (1995, 1997) investigated the long-term returns of the CIR model, and Zhao (2009) extended those results to the jump models.

Noting that the reversion level γ in (1) is a constant, in order to better reflect the time dependence caused by the cyclical nature of the economy or by expectations concerning the future impact of monetary, as in Deelstra and Delbaen (1995, 1997), we can assume that the short-term interest rate model has a stochastic reversion level.

As described above, there is a natural motivation for considering the stochastic interest rate model where all three features, delay, jumps and time dependence of reversion level, are presented. In this paper, we consider the stochastic interest rate model with jumps and memory in the form

$$\begin{cases} dX(t) = \{2\beta X(t) + \delta(t)\}dt + \sigma X^{\gamma}(t-\tau)\sqrt{|X(t)|}dW(t) \\ \qquad\qquad + \int_U g(X(t-), u)\tilde{N}(dt, du), \\ X_0 = \xi \in \mathscr{D}, \end{cases} \tag{2}$$

where $X(t-) := \lim_{s\uparrow t} X(s)$ and $\mathscr{D}$ denotes all real bounded càdlàg functions defined on $[-\tau, 0]$ for some $\tau > 0$. The integral $\int_U g(X(t-), u)\tilde{N}(dt, du)$ depends on the Poisson measure and is regarded as a jump. The diffusion term is dependent on the past through $X^{\gamma}(t-\tau)$ and so is called delay or memory. Precise assumptions on the data of the problem (2) are given in Section 2 below.

The long-term interest rates play an important role in finance and insurance. For instance, the long-term interest rates determine when homeowners refinance their mortgages in mortgage pricing, play a dominant role in life insurance, decide when one should exchange a long bond to a short bond in pricing an option. In this light, for the instantaneous interest rate model (2), it is interesting to investigate the long-term return $t^{-\mu}\int_0^t X(s)ds$ for some $\mu \geq 1$. We shall reveal that the long-term return $t^{-\mu}\int_0^t X(s)ds$ converges almost surely to a stochastic reversion level, which will be stated in Theorem 1 below. As stated in Deelstra and Delbaen (2000), the limit of long-term return $t^{-\mu}\int_0^t X(s)ds$ can be applied in the determination of models of participation in the benefit or of saving products with a guaranteed minimum return.

As we know, one-factor models imply that the instantaneous returns on bonds of all maturities are perfectly correlated, which is clearly inconsistent with reality, e.g., Longstaff and Schwartz (1992). However empirical research, e.g., Brigo and Mercurio (2006), Cassola and Barros Luis (2001) and Longstaff and Schwartz (1992), has suggested that two-factor models, including the short-term interest rate and the instantaneous variance of changes in the short-term interest rate, are better than one-factor models to capture the behavior of the term structure in the real world. This is because the two-factor models allow contingent claim values to reflect both the current level of interest rates as well as the current level of interest rate volatility. Cox et al. (1985, p. 399) introduced a model by two independent factors, r_1 and r_2, and the instantaneous interest rate is the sum of two factors, that is,

$$\begin{cases} r(t) = r_1(t) + r_2(t) \\ dr_i(t) = \kappa_i(\theta_i - r_i(t))dt + \sigma_i(t)\sqrt{r_i(t)}dB_i(t), \quad i = 1, 2, \end{cases}$$

where $B_1(t)$ and $B_2(t)$ are independent Brownian motions, θ_i is the long-term mean factor r_i reverts to, and κ_i and σ_i are constants. In another example of a multi-factor model, the domestic short-rate r_d and the European short-rate r_e satisfy the following stochastic differential equation (SDE)

$$\begin{cases} dr_d = [a + b(r_e - r_d)]dt + \sigma_d dB_d(t), \\ dr_e = c(d - r_e)dt + \sigma_e dB_e(t), \\ \mathrm{Cov}(dB_d(t), \qquad dB_e(t)) = \rho dt, \end{cases}$$

where $B_d(t)$ and $B_e(t)$ are Brownian motions with instantaneous correlation ρ, and $a, b, c, d, \sigma_d, \sigma_e$ are positive constants, Corzo and Schwartz (2000) investigated how to price the European bond and several other interest rate derivatives. As an immediate application of Theorem 1, we consider the long-term return of the two-factor model in the form

$$\begin{cases} dX(t) = \{2\beta_1 X(t) + \delta(t)\}dt + \sigma_1 X^{\gamma_1}(t-\tau)\sqrt{|X(t)|}dW_1(t) \\ \qquad\qquad + \vartheta_1 X(t)\int_U u\tilde{N}_1(dt, du), \\ dY(t) = \{2\beta_2 Y(t) + X(t)\}dt + \sigma_2 Y^{\gamma_2}(t-\tau)\sqrt{|Y(t)|}dW_2(t) \\ \qquad\qquad + \vartheta_2 Y(t)\int_U u\tilde{N}_2(dt, du) \end{cases} \tag{3}$$

with the initial data $(X(t), Y(t)) = (\xi(t), \eta(t)), t \in [-\tau, 0]$. Here $W_1(t), W_2(t)$ are Brownian motions, $N_1(dt, du), N_2(dt, du)$ represent Poisson counting measures, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with characteristic measures $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ respectively, and $\xi, \eta \in \mathscr{D}$. More details on the parameters of model (3) are to be presented in Section 4. In model (3), the short interest rate $Y(t)$ follows an extended CIR model with stochastic reversion level $-X(t)/(2\beta)$, where $X(t)$ follows an extension of the square root process. For model (3), we are interested in the almost sure convergence of the long-term return $t^{-\mu}\int_0^t Y(s)ds$ for some $\mu \geq 1$. Such a convergence can be applied to the finance and insurance markets. For example, the customer wants a return as high as possible, and insurance companies wonder how the percentage of interest should be determined when they promise a certain fixed percentage of interest on their insurance products such as bonds, life-insurance and so on.

The rest of the paper is organized as follows. In Section 2 we introduce some preliminaries, show the nonnegative property of $X(t)$ as nominal instantaneous interest rate determined by (2), and give an auxiliary lemma of Theorem 1. Section 3 is devoted to the almost sure convergence of long-term return $t^{-\mu}\int_0^t X(s)ds$ for some $\mu \geq 1$ with $X(t)$ a generalized CIR model determined by (2), in which the corresponding results in Deelstra and Delbaen (1995, 1997) and Zhao (2009) are extended. In the final section, as an application of Theorem 1, we consider the almost sure convergence of long-term return $t^{-\mu}\int_0^t Y(s)ds$ for some $\mu \geq 1$ determined by (3) with stochastic reversion level $-X(t)/(2\beta)$, where the results of Zhao (2009, Theorem 2) are developed.