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A flexible tree for evaluating guaranteed minimum withdrawal benefits under deferred life annuity contracts with various provisions

Sharon S. Yang [a,b], Tian-Shyr Dai [c,*]

[a] Department of Finance, National Central University, Taiwan

[b] Risk and Insurance Research Center, College of Commerce, National Chengchi University, Taiwan

[c] Department Information and Financial Management and Institute of Finance, National Chiao-Tung University, Taiwan

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ABSTRACT

Valuing guaranteed minimum withdrawal benefit (GMWB) has attracted significant attention from both the academic field and real world financial markets. However, some popular provisions of GMWB contracts, like the deferred life annuity structure, rollup interest rate guarantees, and surrender options are hard to be evaluated analytically and are rarely addressed in the academic literature. This paper proposes a flexible tree model that can accurately evaluate the values and the fair insurance fees of GMWBs. The flexibility of our tree allows us to faithfully implement the aforementioned provisions without introducing significant numerical pricing errors. The mortality risk can also be easily incorporated into our pricing model. Our numerical results verify the robustness of our tree and demonstrate how the aforementioned provisions and the mortality risk significantly influence the values and the fair insurance fees of GMWBs.

1. Introduction

The variable annuity (VA)[1] is a popular insurance product sold in the U.S. retirement market. When people purchase a VA product, they either pay a lump sum or make periodic payments into a fund that is invested in an investment portfolio, such as a mutual fund. The account value of the fund accumulates in accordance with the performance of the investment portfolio. Policyholders can choose the investment portfolio and thus bear the investment risk. In recent years, granting the investment guarantee has become a popular design with VA products. With this design, the insurer guarantees a specified return on the policy's account value through various types of investment guarantees, such as guaranteed minimum death benefits (GMDBs), guaranteed minimum maturity benefits (GMMBs), guaranteed minimum income benefits (GMIBs), and guaranteed minimum withdrawal benefits (GMWBs). To refer to this broad class of guarantees, we employ the term GMXBs and note that, regardless of the type, the guarantee features of GMXBs provide downside risk protection to policyholders. These VA products have enjoyed great market success in the United States and Asia. Condron (2008) suggests that the guarantee features account for the growing popularity of VA, as manifested in more than \$1.35 trillion currently invested, a 50% increase over the previous five years. The products are also gaining popularity in international markets (Ledlie et al., 2008).

Granting GMXBs means that VA products contain embedded financial options. The various guarantees can be viewed as various types of exotic options, and the pricing of these exotic options has become a critical research focus. Brennan and Schwartz (1976, 1979) first priced unit-linked contracts with an asset guarantee (i.e., GMMB). The payoff of a GMMB is similar to that of an ordinary European option, so they derived closed-form pricing formulas by taking advantage of classic Black–Scholes assumptions. Milevsky and Posner (2001) regarded GMDB benefits as a Titanic option and presented closed-form solutions with a simplified exponential mortality model. Analytic solutions for valuing GMIBs appear in Boyle and Hardy (2003) and Ballotta and Haberman (2003, 2006). Among these investment guarantees, the GMWB guarantee has, however, attracted particularly significant attention and sales in recent years. A GMWB contract allows the policyholder to withdraw funds periodically for a contractually specified amount for a specified guaranteed withdrawal period, regardless of the performance of the underlying investment portfolio. When the contract expires, the holder can either redeem the remaining investment or convert it into a life annuity. Recent research thus addresses the pricing of GMWB contracts, starting with Milevsky and Salisbury (2006), who first introduce the concept of a Quanto Asian put for valuing GMWBs. Chen et al. (2008) then consider the

* Corresponding author.
E-mail addresses: syang@ncu.edu.tw (S.S. Yang), d88006@csie.ntu.edu.tw, cameldai@mail.nctu.edu.tw (T.-S. Dai).

[1] Also known as unit-linked products in the United Kingdom.

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jump effect and employ a jump diffusion process to value GMWBs. Finally, Dai et al. (2008) instead provide a rigorous derivation of the singular stochastic control model for pricing variable annuities with GMWBs using the Hamilton–Jacobi–Bellman (HJB) equation. Bauer et al. (2008) also consider a universal pricing framework in which they can price various GMXBs consistently using simulation techniques.

Because the insurance policy entitles policyholders to terminate their contracts before the maturity date and receive a certain cash refund (called the surrender value), taking the surrender feature into account has become a mainstream tactic for valuing the equity-linked policies. Shen and Xu (2005) study fair valuations of equity-linked policies with interest rate guarantees in the presence of surrender options. Costabile et al. (2008) consider fair periodical premiums for equity-linked policies with a surrender option under a binomial model, and then tackle the problem of computing fair periodical premiums for an equity-linked policy with a maturity guarantee and an embedded surrender option. Regarding the recently developed GMWB contract, since its payoff is more complex than that of other guarantee types, it turns out that valuing a GMWB contract is much more difficult, especially if surrender is allowed. Milevsky and Salisbury (2006) assume that an optimal withdrawal policy seeks to maximize the annuity value by lapsing the product at an optimal time. Our paper extends their work by analyzing how the policyholders optimize their surrender decisions in an effort to strike a balance among losses of the time value due to delayed withdrawal, losses due to mortality risk, and early redemption penalties. Much of the literature has been concerned with the optimal withdrawal behavior as opposed to the surrender options. Our tree can be extended to model the optimal withdrawal without difficulty.

Most studies oversimplify the various provisions of the GMWB contract in order to make their pricing models tractable. However, these oversimplifications might result in significant pricing deviations as illustrated in the numerical experiments in Section 4. For example, most GMWBs are associated with deferred variable annuities, and guaranteed withdrawals normally take place after deferred periods. Different guaranteed withdrawal amounts might be designed for a deferred life annuity, such as a rollup interest rate guarantee. To the best of our knowledge, the existing literature assumes that the guaranteed withdrawal starts immediately, at the inception of the policy, even though this discrepancy could result in significantly different pricing results. Therefore, we investigate the effect of deferred periods and various guarantee designs on the fair charge numerically. Besides, the mortality improvements in recent years can affect the value of GMWBs. We also consider mortality improvements when valuing the GMWB contracts by incorporating mortality improvement factors into our tree model. In addition, we also analyze whether the presence of mortality risk, rollup interest rate guarantees, the volatility of the underlying investment, and the redemption penalty influence the value of surrender options.

Evaluating the fair charge for granting the GMWB is the key goal for the GMWB evaluation problem. This insurance fee is subtracted from the account value in return for the investment guarantee and provisions provided by the insurance company. Note that the value to hold a GMWB contract, abbreviated as the "value of the GMWB" for simplicity, decreases with the increment of the insurance fee. The fair charge is the fee that makes the value of the GMWB equal to the policyholder's initial investment. Complex provisions of GMWB contracts and the mortality risk prevent the fair charge from being analytically solved. On the other hand, evaluating the values of complex GMWB contracts with numerical methods could generate oscillating pricing results, which would result in no or multiple solutions for the fair charge.

The major contribution of this paper is that it develops an accurate numerical tree method to calculate the value of the GMWB and the fair charge. The flexible nature of the tree can help us to incorporate the mortality models into the tree, to combine the evolution of different account value processes during the deferred and the withdrawal periods, and to deal with optimal surrender and withdrawal decisions. In addition, to faithfully model various provisions of GMWB contracts without incurring significant numerical pricing errors, our tree also borrows the trinomial structures proposed by the stair tree (Dai, 2009) and the bino-trinomial tree (BTT; Dai and Lyuu, 2010). The stair tree uses trinomial structures to faithfully model the downward jump of stock prices due to discrete dividend payouts; this idea can be used to model the downward jump in the account value of GMWB contracts due to discrete withdrawal and the fair charge. It can also help us to adjust the tree structure in order to price GMWB more stably. To alleviate the price oscillation problem due to the nonlinearity errors (Figlewski and Gao, 1999), the BTT uses the trinomial structure to adjust the tree structure to coincide with "critical locations"—the locations where the function of the financial derivative value is highly nonlinear. This paper also uses the trinomial structure to make the tree coincide with certain critical locations caused by the periodical withdrawal guarantees listed in the GMWB contracts. Thus our tree can stably price the value of GMWBs without numerical errors and can thus find the fair charge stably.

The structure of this paper is as follows. In Section 2, we describe the GMWB contract and some important provisions, for instance, the deferred life annuity structure and the rollup interest rate guarantee design. The process of the GMWB account value and the payoff of the policyholder are then modeled according to the provisions. The required knowledge of the tree model is also reviewed in the same section. In Section 3, we construct a new tree model and implement the backward induction method to deal with the valuation of complicated provisions (e.g., the deferred annuity structure, rollup interest rate guarantees, and surrender options) and mortality risk in GMWB contracts. The structure of our tree is sophisticatedly designed to suppress numerical pricing errors in order to generate the stable value of GMWBs and the fair charge. The numerical results in Section 4 analyze how the presence of different provisions influences the fair charge and GMWB values. Section 5 concludes the paper.

2. The structure of the GMWB contract and tree models

2.1. *Account dynamics of the GMWB contract with a deferred variable annuity*

We assume a single-premium deferred variable annuity associated with the GMWB. The policyholder deposits an initial premium ω_0 in an account that is invested in a selected fund portfolio and is guaranteed the right to withdraw a specified amount from that account at each withdrawal date during the guaranteed withdrawal period. Let W_t denote the account value at time t for the GMWB contract, and initial account value $W_0 = \omega_0$. The account value changes according to the return on the invested fund portfolio and diminishes by the periodical withdrawals and payments of the insurance fee. Let the time intervals $[0, T_1]$ and $[T_1, T_2]$ denote the deferred period and withdrawal period, respectively, whereas T_2 is the maturity date. The policyholders are only allowed to make guaranteed withdrawals periodically during the withdrawal period. During the deferred period, the account value changes only in relation to the return on the underlying asset and the insurance fee. Thus the stochastic differential equation (SDE) of W_t during the deferred period is

$$dW_t = (r - \alpha)\, W_t dt + \sigma W_t dB_t, \quad 0 < t \le T_1, \tag{1}$$