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Pricing European options on deferred annuities

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ABSTRACT

This paper considers the pricing of European call options written on pure endowment and deferred life annuity contracts, also known as guaranteed annuity options. These contracts provide a guaranteed value at the maturity of the option. The contract valuation is dependent on the stochastic interest rate and mortality processes. We assume single-factor stochastic square-root processes for both the interest rate and mortality intensity, with mortality being a time-inhomogeneous process. We then derive the pricing partial differential equation (PDE) and the corresponding transition density PDE for options written on the pure endowment and deferred annuity contracts. The general solution of the pricing PDE is derived as a function of the transition density function. We solve the transition density PDE by first transforming it to a system of characteristic PDEs using Laplace transform techniques and then applying the method of characteristics. Once an explicit expression for the density function is found, we then use sparse grid quadrature techniques to generate European call option prices on the pure endowment and deferred annuity contracts. This approach can easily be generalised to other contracts which are driven by similar stochastic processes presented in this paper. We test the sensitivity of the option prices by varying independent parameters in our model. As option maturity increases, the corresponding option prices significantly increase. The effect of mispricing the guaranteed annuity value is analysed, as is the benefit of replacing the whole-life annuity with a term annuity to remove volatility of the old age population.

1. Introduction

In this paper we derive techniques for pricing deferred annuity and pure endowment options that can be used for managing longevity risk in life insurance and annuity providers' portfolios. Analytical techniques for deriving the joint interest and mortality rate probability density function are drawn from Chiarella and Ziveyi (2011), where the dynamics of the underlying security evolve under the influence of stochastic volatility. Given our time-inhomogeneous mortality process, we use sparse grid quadrature methods to solve option prices under the risk-neutral measure.

Insurance companies and annuity providers are increasingly exposed to the risk of ever improving mortality trends across all ages, with a greater portion of survivors living beyond 100 years (Carriere (1994), Currie et al. (2004) and CMI (2005) among others). Such mortality improvements, coupled with the unavailability of suitable hedging instruments, pose significant challenges to annuity providers seizing from the risk of longer periods of annuity payments than initially expected. At present mortality risk is non-tradable (Blake et al., 2006) and there is no market to hedge these risks other than reinsurance. Blake and Burrows (2001) highlight that annuity providers have been trying to hedge mortality risk using costly means such as the construction of hedged portfolios of long-term bonds (with no mortality risk). The hedging of both the interest rate and longevity risks is important to annuity providers, and the inability to purchase long-term bonds also hinders the annuity providers ability to hedge the interest rate risk, although there are a greater range of interest rate hedging methods and instruments than those for longevity risk. In this paper we focus on longevity risk. Due to the long-term nature of life annuity contracts, the development of an active market where longevity risk can easily be priced, traded and hedged, requires more advanced pricing and hedging methods. A number of securities that can make up this market have already been proposed in the literature, and these include longevity bonds, mortality derivatives, securitised products among others (Bauer, 2006b; Blake et al., 2006).

Blake et al. (2006) and Bauer (2006a,b) demonstrate that, if mortality risk can be traded through securities such as longevity bonds and swaps, then the techniques developed in financial markets can be adapted and implemented for mortality risk. A number of papers have also developed models for pricing guaranteed annuity options. Milevsky and Promislow (2001) develop algorithms for valuing mortality contingent claims by taking the underlying securities as defaultable coupon paying bonds with the time of death

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as a stopping time. Boyle and Hardy (2003) use the numeraire approach to value options written on guaranteed annuities. They detail the challenges experienced in the UK where long-dated and low guaranteed rates were provided relative to the high prevailing interest rates in the 1970s. Interest rates fell significantly in the 1990s, leading to sharp increases in the value of guaranteed contracts, and this had a significant impact on annuity providers' profitability.

A significant number of empirical studies have been presented showing that mortality trends are generally improving and the future development of mortality rates are uncertain and require stochastic models. Proposed stochastic models include Milevsky and Promislow (2001), Dahl et al. (2008) and Biffis (2005). A number of stochastic mortality models have been motivated by interest rate term-structure modelling literature (Cox et al., 1985; Dahl et al., 2008; Litterman and Scheinkman, 1991) as well as stochastic volatility models such as that proposed in Heston (1993).

The main aim of this paper is to devise a novel numerical approach for the pricing and hedging of deferred mortality contingent claims with special emphasis on pure endowment options and deferred immediate annuity options. We start off by devising techniques for pricing deferred insurance contracts, which are the underlying assets. Analytical solutions can be derived for pure endowment contracts using the forward measure approach; however, this is not possible for deferred immediate annuities where analytical approximation techniques have mostly been used as in Singleton and Umantsev (2002) when valuing options on coupon paying bonds. Having devised models for the underlying securities, we then present techniques for valuing European style options written on these contracts.

We assume that the interest rate dynamics is driven by a single-factor stochastic square-root process, while the time-inhomogeneous mortality dynamics is a one-factor version of the model proposed in Biffis (2005). The long-term mean reversion level of the mortality process is a time-varying function following the Weibull mortality law. This provides a reference mortality level for each age in a cohort. The model definition guarantees positive mortality rates. Although the mortality intensity can fall below our reference rate, careful selection of the parameters limits this occurring.

We use hedging arguments and Ito's Lemma to derive a partial differential equation (PDE) for options written on the deferred insurance contracts. We also present the backward Kolmogorov PDE satisfied by the two stochastic processes under consideration. We present the general solution of the pricing PDE by using Duhamel's principle. The solution is a function of the joint probability density function, which is also the solution of the Kolmogorov PDE. We solve the transition density PDE with the aid of Laplace transform techniques, thereby obtaining an explicit expression for the joint transition density function. Using the explicit density function, we then use sparse grid quadrature methods to price options on deferred insurance contracts.

While Monte Carlo simulation techniques are effective for option pricing, the analytic solution of the joint density function provides valuable insight into longevity risk. From the analytic solution one can easily derive expressions for the hedge ratios, such as the option price sensitivity with respect to the interest rate and mortality processes. The framework can be extended to multiple interest and mortality risk factors, with only a small increase in computation requirements.

The remainder of this paper is organised as follows: Section 2 presents the modelling framework for the interest rate and mortality rate processes. We then provide the option pricing framework in Section 3. It is in this section where we derive the option pricing PDE and the corresponding backward Kolmogorov PDE for the density function. With the general solution of the pricing PDE presented, we then outline a step-by-step approach for solving the transition density PDE using Laplace transform techniques. The explicit expressions for the deferred pure endowment and deferred annuity contracts together with their corresponding option prices are presented in Section 4. All numerical results are presented in Section 5. Section 6 concludes the paper. Where appropriate, the derivations and proofs are included to the appendices.

2. Modelling framework

The intensity-based modelling of credit risky securities has a number of parallels with mortality modelling (Lando, 1998; Biffis, 2005). We are interested in the first stopping time, τ, of the intensity process $\mu(t;x)$, for a person aged x at time zero. Starting with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability measure. The information at time-t is given by $\mathbb{F} = \mathbb{G} \vee \mathbb{H}$. The sub-filtration $\mathbb{G}$ contains all financial and actuarial information except the actual time of death.

There are two $\mathbb{G}$-adapted short-rate processes, $r(t)$ and $\mu(t;x)$, representing the instantaneous interest and mortality processes respectively. The sub-filtration $\mathbb{H}$ is the σ-algebra with death information. Let $N(t) := \mathbb{1}_{\tau \le t}$ be an indicator function, if the compensator $A(t) = \int_0^t \mu(s;x)ds$ is a predictable process of $N(t)$, then $dM(t) = dN(t) - dA(t)$ is a $\mathbb{P}$-martingale, where $dA(t) = \mu(t;x)dt$.

There also exists another measure where $dM(t)$ is a $\mathbb{Q}$-martingale, under which the compensator becomes $dA(t) = \mu^{\mathbb{Q}}(t;x)dt$, with $\mu^{\mathbb{Q}}(t;x) = (1+\phi(t))\mu(t;x)$ and $\phi(t) \ge -1$. The predictable process $\phi(t)$ represents a market price of idiosyncratic, or individual, mortality risk of the insurance contract. The idiosyncratic risk is important to the annuity provider, but for the market this risk can be diversified, and we assume $\phi(t) = 0$ in this paper. In the absence of market data, we make the usual simplifying assumption that under the $\mathbb{Q}$-measure interest and mortality rates are independent; we do not assume a change in the economic conditions to affect the risk premium in longevity securities.

Proposition 2.1. *In the absence of arbitrage opportunities there exists an equivalent martingale measure $\mathbb{Q}$ where $C(t, T, r, \mu; x)$ is the t-value of an option contract with a pay-off function, $P(T, r, \mu; x)$, at time-T. The payment of $P(T, r, \mu; x)$, which is a $\mathbb{G}$-adapted process, is conditional on survival to the start of the period T, otherwise the value is zero. The time t-value of the option can be represented as*

$$\begin{aligned} C(t,T,r,\mu;x) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} P(T,r,\mu;x)\mathbb{1}_{\tau>T} \middle| \mathcal{F}_t\right] \\ &= 1_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T [r(u)+\mu(u;x)]du} P(T,r,\mu;x) \middle| \mathcal{G}_t\right]. \end{aligned} \tag{2.1}$$

Proof. The law of iterated expectations can be used to show this; see Bielecki and Rutkowski (2002) or Biffis (2005) for detail. □

In our framework, time-T is always the option maturity age. If $P(T, r, \mu; x) \equiv 1$, then our contract resembles a credit risky zero coupon bond. In actuarial terms, this is a pure endowment contract written at time-t that receives 1 at time T if the holder is still alive. If $P(T, r, \mu; x)$ is the value of a stream of payments starting at T, conditional on survival, then we are pricing a deferred immediate annuity. The contract value, $P(T, r, \mu; x)$, can also take the form of an option pay-off. In this scenario, the strike price, K, represents a guaranteed value at time-T on an endowment or annuity contract.

One approach to solving Eq. (2.1) is to use a forward measure approach. If we use $P(T, r, \mu; x)$ as numeraire, we can rewrite (2.1) as

$$C(t,T,r,\mu;x) = 1_{\tau>t} P(t,r,\mu;x)\mathbb{E}^{Q^P}\left[P(T,r,\mu;x)|\mathcal{F}_t\right] \tag{2.2}$$