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Extremes and products of multivariate AC-product risks

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ABSTRACT

With motivation from Tang et al. (2011), in this paper we consider a tractable multivariate risk structure which includes the Sarmanov dependence structure as a special case. We derive several asymptotic results for both the sum and the product of such risk and then present three applications related to actuarial mathematics.

1. Introduction

Many modern actuarial tasks such as quantification of large risks and aggregated risk, estimation of ruin probabilities in the presence of financial risks, or reinsurance pricing accounting for both claims and expenses strongly rely on the use of multivariate extreme value theory. Typically, the adequacy of the probabilistic models employed by the actuaries is determined by their flexibility to allow for the dependence among risks. Most of classical insurance models assume independence of risks, a phenomenon which is rarely observed in practical actuarial tasks. The role of the dependence among risks is crucial, especially when modelling the impact of large risks. Dependence modelling and in particular that of large risks has been the topic of several contributions such as Goovaerts et al. (2005), Denuit et al. (2006), Li et al. (2010), Asimit et al. (2011), Chen (2011), Haug et al. (2011), Kortschak (2011), Manner and Segers (2011), Tang et al. (2011), and Chen and Yuen (2009, 2012) among many others.

Asimit et al. (2011) successfully demonstrates the role of asymptotic dependence and asymptotic independence in actuarial modelling. As shown therein, multivariate risks which exhibit asymptotic dependence imply in general different results compared to multivariate risks which have asymptotic independent components. Tractable multivariate distributions like the Fairlie–Gumbel–Morgenstern (FGM) ones exhibit asymptotic independence.

In various risk models employed by actuaries two related tasks are the asymptotic analysis of aggregated risk, and the asymptotic quantification of the effect random scaling (or deflation) of risks. Since the empirical data always support the fact that risks are stochastically dependent, aggregation of dependent risks has become recently a key topic for insurance, finance, and risk management. Recent results of Mitra and Resncik (2009) and Asimit et al. (2011) pave the way for the analysis of the impact of a single large risk to the aggregated risk.

In a mathematical framework, if $X_0, \ldots, X_n$ are non-negative random variables (rv's) with distribution functions (df's) $F_0, \ldots, F_n$, then the aggregated risk is $S = \sum_{i=0}^{n} X_i$. In order to avoid triviality, we assume that the risks are all non-degenerate at zero. Large values of S mean large financial risks for the company, and therefore the actuarial interest focuses mainly on the quantification of the probability of such large values, i.e., $\mathbb{P}(S > u)$ where the level u reaches some extreme point.

In another context, X_0 can be considered as the base risk, whereas $X_1, \ldots, X_n$ as random deflators/inflators. Of actuarial interest is the asymptotic tail behaviour of the ultimate deflated

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risk ($u \to \infty$)

$$\mathbb{P}(Z > u), \quad \text{with } Z = X_0 \prod_{i=1}^{n} X_i. \tag{1.1}$$

For independent risks recent results in this direction are derived in Hashorva et al. (2010).

The main goal of this paper is to introduce a tractable class of dependent risks which allows for explicit calculation of various actuarial quantities of interest. The motivation for introducing such a class of risks comes from the simple structure of multivariate FGM df's. By definition, a $(n+1)$-dimensional random vector $\boldsymbol{X} = (X_0, \ldots, X_n)$ has a multivariate FGM df Q with marginal df's $F_0, \ldots, F_n$ if

$$Q(x_0, \ldots, x_n) = \prod_{i=0}^{n} F_i(x) \left[1 + \sum_{0 \le i < j \le n} \theta_{ij} \overline{F_i}(x_i) \overline{F_j}(x_j) \right],$$
$$x_i \in [0, \hat{x}_i], \ 0 \le i \le n, \tag{1.2}$$

where $\overline{F_i} := 1 - F_i$, and θ_{ij}'s are some real constants which satisfy certain restrictions so that Q is a df. Here $\hat{x}_i := \sup\{x \in \mathbb{R} : F_i(x) < 1\}$ stands for the upper endpoint of the marginal df F_i.

Throughout the paper we assume that the risks are non-negative, thus F_i has support on $[0, \infty)$.

The tractability of $\boldsymbol{X}$ with df Q given by (1.2) relates to the fact that Q is obtained by the product distribution $Q^* = \prod_{i=0}^{n} F_i$ $\left(\text{in fact } Q^*(x_0, \ldots, x_n) = \prod_{i=0}^{n} F_i(x_i)\right)$. By a closer inspection, it follows that

$$Q(x_0, \ldots, x_n) = \int_0^{x_0} \cdots \int_0^{x_n} \left[1 + \sum_{0 \le i < j \le n} \theta_{ij} (1 - 2F_i(s_i))(1 - 2F_j(s_j)) \right] \times Q^*(\mathrm{d}s_0, \ldots, \mathrm{d}s_n)$$

holds for any $x_i \in [0, \hat{x}_i]$, $0 \le i \le n$. The larger class of multivariate Sarmanov distributions is introduced by substituting above $1 - 2F_i$ by some kernel ϕ_i; some insurance applications of Sarmanov distributions are illustrated in Tang et al. (2011) and Yang and Wang (2012).

Motivated by the underlying relationship between Q and the product df $Q^* = \prod_{i=0}^{n} F_i$, in this paper we consider a wider class of multivariate df's which are absolutely continuous with respect to a product df—we refer to that as AC-product class. Specifically, the members of this class are all absolutely continuous df's with respect to Q^*.

It turns out that under some weak conditions the asymptotic behaviour of the aggregated risk S and the deflated risk Z for risks with an AC-product distribution can be derived explicitly.

Organisation of the rest of the paper: In Section 2 we briefly discuss some basic properties of AC-product distributions. Further, we derive a novel result concerning the Sarmanov distribution, which is the canonical example of the AC-product class. Section 3 shows the asymptotic independence of AC-product risks, whereas Section 4 investigates the asymptotic behaviour of the deflated risk Z under extreme value type conditions on the marginal df's. In Section 5 we present three applications concerning risk aggregation, Value-at-Risk and conditional tail expectation, and the probability of ruin under risky investment. The proofs of all the results are postponed to Section 6.

2. Multivariate AC-product and Sarmanov distributions

In this section we present some details on the class of AC-product distributions and Sarmanov distributions. Hereafter $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$ is a $(n+1)$-dimensional random vector with non-negative univariate marginal df's F_i, $0 \le i \le n$. It is not standard to write the first component of $\boldsymbol{X}$ by X_0; we do this since this component will be a reference one in the part when the products of the components of $\boldsymbol{X}$ are discussed. Clearly, if $\boldsymbol{X}$ possesses the df $Q^* = \prod_{i=0}^{n} F_i$, then the random vector $\boldsymbol{X}$ has independent components, a situation which is often not encountered in practical applications. Starting from this independence setup, a tractable dependence structure is introduced by considering $\boldsymbol{X}$ such that its df Q is absolutely continuous with respect to the product df Q^* i.e.,

$$Q(x_0, \ldots, x_n) = \int_0^{x_0} \cdots \int_0^{x_n} \eta(s_0, \ldots, s_n) \, Q^*(\mathrm{d}s_0, \ldots, \mathrm{d}s_n),$$
$$x_i \in [0, \hat{x}_i], \ 0 \le i \le n, \tag{2.1}$$

where $\eta(\cdot)$ is a non-negative measurable function, i.e., if we write (2.1) as

$$\mathrm{d}Q = \eta \cdot \mathrm{d}Q^*,$$

we see that η is the Radon–Nikodym derivative. Throughout this paper

$$X_0^*, \ldots, X_n^*$$

are independent rv's with df's F_i, $0 \le i \le n$, respectively, and thus joint df Q^*. We refer to Q as an AC-product distribution. Since Q is a proper df we shall assume that

$$\mathbb{E}\left\{\eta(X_{x_0}^*, \ldots, X_{x_n}^*)\right\} < \infty \tag{2.2}$$

almost surely with respect to Q^* where $X_{x_i}^* = X_i^*$ or $X_{x_i}^* = x_i$ with x_i in the support of F_i. Further, we suppose that

$$\mathbb{E}\left\{\eta(X_0^*, \ldots, X_n^*)\right\} = 1 \tag{2.3}$$

holds. Clearly, (2.2) is satisfied when $\eta(\cdot)$ is a bounded function.

The Sarmanov distributions mentioned in the Introduction are obtained when

$$\eta(x_0, \ldots, x_n) = 1 + \sum_{0 \le k < l \le n} \theta_{kl} \phi_k(x_k) \phi_l(x_l), \tag{2.4}$$

with $\phi_0, \ldots, \phi_n$ some given real-valued kernels, and θ_{kl}, $0 \le k < l \le n$ non-negative constants.

In order for such $\eta(\cdot)$ to define a proper df, we shall impose the following assumptions on the kernels:

A1. ϕ_i, $0 \le i \le n$ are not identical to 0 in $[0, \hat{x}_i]$;

A2. for all $x_i \in [0, \hat{x}_i]$, $0 \le i \le n$ we have

$$\sum_{0 \le k < l \le n} \theta_{kl} \phi_k(x_k) \phi_l(x_l) \ge -1 \tag{2.5}$$

almost surely with respect to Q^*;

A3. for any $0 \le i \le n$ we have

$$\mathbb{E}\{\phi_i(X_i)\} = 0. \tag{2.6}$$

Apart form the choice $\phi_i = 1 - 2F_i$ which leads to the FGM distribution, another common specification of the kernels is $\phi_i(s) = g_i(s) - \mathbb{E}\{g_i(X_i)\}$, $s > 0$, for some function g_i such that $\mathbb{E}\{g_i(X_i)\} < \infty$.

We may consider for instance $g_i(s) = \exp(-s)$, or $g_i(s) = s^{\alpha_i}$, $\alpha_i \in \mathbb{R}$, provided that $\mathbb{E}\left\{X_i^{\alpha_i}\right\} < \infty$ and $\hat{x}_i < \infty$. The next lemma shows that the kernels need to obey certain asymptotic restrictions.