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Optimal decision on dynamic insurance price and investment portfolio of an insurer

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ABSTRACT

We establish a model of insurance pricing with the assumption that the insurance price, insurer investment returns, and insured losses are correlated stochastic processes. We consider the effect of demand on price where the objective of the pricing model is to maximize the expected utility of the insurer's terminal wealth. Based on a Hamilton–Jacobi–Bellman (HJB) equation, we simultaneously solve for the optimal price of an insurance contract and the optimal investment portfolio of an insurer. The results show that quantity demanded of insurance contracts affects the optimal allocation to risky assets in the insurer's investment portfolio. Our results also show that the drift and volatility of the insurance price process will affect the investment strategy, in addition to the effect of the drift and volatility of the investment process itself.

1. Introduction

The literature on investment strategy for an insurance firm is quite extensive. Yang and Zhang (2005) examined the optimal investment policies of an insurer with jump-diffusion risk process. They obtained the closed form expression of the optimal investment policy for an exponential utility function. These authors also analyzed the insurer's optimal policy for general objective function. Lin and Li (2011) considered an optimal reinsurance-investment problem of an insurer whose surplus process follows a jump-diffusion model. The dynamics of the risky asset are governed by a constant elasticity of variance model to incorporate conditional heteroscedasticity. The objective of the insurer is to choose an optimal insurance-investment strategy so as to maximize the expected exponential utility of terminal wealth.[1]

The extant literature addresses the issue of finding the optimal insurer investment portfolio under the assumption that the insurance price is given and is expressed per unit of exposure. However, in reality the risks and returns of different investment portfolios vary, and portfolio variation may affect the price of insurance. Additionally, the demand for insurance will affect the insurance price and, under some circumstances, the risk of premium volume written may balance investment risk.

Much work in this area also has focused on optimal insurance pricing. Taylor (1986) developed a theory in which the premium rate may be optimally determined by accounting for the effects of competition. Emms and Haberman (2005) examined the calculation of premiums using optimal control theory by maximizing the terminal wealth of an insurer, considering the insurance demand function. Emms (2007a) extended his research

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[1] Other related research works include Browne (1995), Hipp and Plum (2000), Liu and Yang (2004), Browne (2000), Korn (2005), Korn and Seifried (2009), Mataramvura and Âksendal (2008), Promislow and Young (2005), Zhang and Siu (2009), Cao and Wan (2009) and Luo and Yang (2010).

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by calculating the premium using dynamic programming with the objective of maximizing the expected utility of the insurer's terminal wealth. Emms (2007b) discussed how to use deterministic control theory to find the optimal premium strategy for an insurer with the constraint of a solvency requirement or a bounded premium. Emms et al. (2007) investigated optimal premium pricing policies in a competitive insurance environment using approximation method and simulation of sample paths. They assumed that the market average price is a diffusion process with the premium as a control function and with the objective of maximizing expected total wealth, over a finite time horizon. Emms and Haberman (2009) determined the optimal premium strategy in a competitive market using a deterministic general insurance model.

Neither did they consider issues such as the effect of different investment strategies on the price of insurance; the effect of the drift and volatility of insurance price, treated as a stochastic process, on insurer investment strategy; or the effect of increasing the number of policies written as a hedge for insurer investment risk. These issues are important in that investment strategy may affect the benefit of an insurer as well as the price of the insurance contracts. Further, the price strategies taken by the insurer likely affect the insurer's investment strategy as well as the benefits of both the customer and the insurer.

Josa-Fombellida and Rincón-Zapatero (2010) studied the optimal management of an aggregated pension fund of defined benefit type. They extended their previous work (Josa-Fombellida and Rincón-Zapatero, 2001, 2004, 2006, 2008a,b) by integrating three uncertain factors including: (1) the fund asset returns; (2) the instantaneous risk-ness rate of interest rate; and (3) the evolution of benefits, based on the growth of salaries and/or other main components of the pension plan with the objective of minimizing deviations of the unfunded actuarial liability from zero along a finite time horizon.

In this article, we extend previous research by considering the mutual effect between price and investment strategy and simultaneously determining optimal price and optimal investment strategy for an insurer. We assume that the average market price of insurance, the return on risky investments and the insured loss are correlated, and we consider the role of the quantity demanded of insurance contracts in decreasing the underwriting risk and hedging the risk of investment. We rely on the pioneering work of Merton (1971) for studying the portfolio problem in continuous time as we consider a pricing model with the objective of maximizing the expected utility of terminal wealth of the insurer. We construct a Hamilton–Jacobi–Bellman (HJB) equation and determine the optimal price of insurance and the optimal investment strategy simultaneously by solving this HJB equation.

The remainder of this paper is organized as follows. Section 2 presents the insurance models. In Section 3, the HJB equation is given and the optimal price and investment portfolio of the insurer are determined. Section 4 shows the sensitivity analysis, and concluding remarks are given in Section 5.

2. The insurance models

We assume that the insurance firm considered can invest its assets in one risk-free asset and one risky asset, which can be traded continuously over time without transaction costs or taxes. Assume that the price process of this risky investment follows a geometric Brownian motion, i.e., $S(t)$ satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_1(t), \qquad S(0) = S_0, \tag{1}$$

where μ is the drift and σ is the volatility of the process. The price of the risk-free asset is assumed to evolve according to

$$dB(t) = rB(t)dt, \tag{2}$$

where r is the continuously compounded risk-free interest rate, with $r < \mu$. We also assume that the claims process can be described by the following stochastic process (see Promislow and Young, 2005):

$$dC(t) = p_1 dt - \sigma_D dW_2(t), \tag{3}$$

where $C(t)$ is the rate of claim loss, p_1 is the average rate of claim loss, and σ_D is the standard deviation of claim loss rate. Then, the surplus process (not including investment) of a specific insurance company is

$$dR(t) = (p(t) - p_1)\, dt + \sigma_D dW_2(t), \tag{4}$$

where $R(t)$ is the surplus of an insurer at time t and p is the price of insurance contracts. In a manner similar to Emms et al. (2007), we also assume that the market average price, $\bar{p}(t)$, follows a geometric Brownian motion satisfying

$$d\bar{p}(t) = \bar{p}(t)\left(\mu_p dt + \sigma_p dW_3(t)\right), \tag{5}$$

where μ_p and σ_p are appropriate drift and volatility parameters. Assume that the price p for insurance charged by a specific company is determined from the value of $\bar{p}$ determined by the stochastic prices above via

$$p(t) = k(t)\bar{p}(t), \tag{6}$$

with $k(t)$ being a company-specific parameter, referred to as the relative price.

The prices of insurance contracts affect the quantity demanded, and the relationship is expressed by the demand function. Taylor (1986) considers two different demand functions: exponential demand function and a constant price elasticity demand function. We will focus on the second function, that is,

$$q(t) = q(0)k^{-at}, \tag{7}$$

where a is a constant related to the elasticity of demand and $q(t)$ is the quantity demanded of an insurer's insurance contracts.

Let $\{X(t) : t \in [0, T]\}$ be the wealth process of the insurance company considered. The wealth process of the insurer issuing the number of insurance contracts, $q(t)$, at time t can be described by the following stochastic differential equations:

$$\begin{aligned} dX(t) &= (p(t)q(t) + \pi(t)(\mu - r) + X(t)r - p_1 q(t))\, dt \\ &\quad + \sigma\pi(t)dW_1(t) + \sigma_D\sqrt{q(t)}dW_2(t) \\ d\bar{p}(t) &= \bar{p}(t)\left(\mu_p dt + \sigma_p dW_3(t)\right) \\ p(t) &= k(t)\bar{p}(t) \\ X(t_0) &= x \end{aligned} \tag{8}$$

where $\{W_1(t), W_2(t), W_3(t) | \mathcal{F}_t, t \geq 0\}$ are three correlated standard Brownian motions on a filtered probability space, and $\mathcal{F}_t$ is the P-augmentation of the natural filtration. Furthermore, the correlation matrix is $\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix}$, and $\pi(t)$ is the allocated amount of the investment portfolio to the risky investment at time t. We assume that $\pi(t)$ is allowed to be less than zero, i.e., short selling is permitted. We also assume that borrowing money is allowed, i.e., $\pi(t)$ can be larger than $X(t)$, with the risk-free rate being the borrowing cost. Please note that the standard deviation of the insured loss for the quantity demanded of insurance contracts, $q(t)$, is $\sqrt{q(t)}\sigma_D$, because $\mathrm{Var}\left(\sum_{i=1}^{q(t)} Y_i\right) = q(t)\sigma_D^2$, where Y_i is the random variable of insured loss for i-th policy and we assume that insurance contracts are independent of each other.