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The connection between distortion risk measures and ordered weighted averaging operators[☆]

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ABSTRACT

Distortion risk measures summarize the risk of a loss distribution by means of a single value. In fuzzy systems, the Ordered Weighted Averaging (OWA) and Weighted Ordered Weighted Averaging (WOWA) operators are used to aggregate a large number of fuzzy rules into a single value. We show that these concepts can be derived from the Choquet integral, and then the mathematical relationship between distortion risk measures and the OWA and WOWA operators for discrete and finite random variables is presented. This connection offers a new interpretation of distortion risk measures and, in particular, Value-at-Risk and Tail Value-at-Risk can be understood from an aggregation operator perspective. The theoretical results are illustrated in an example and the degree of orness concept is discussed.

1. Introduction

The relationship between two different worlds, namely risk measurement and fuzzy systems, is investigated in this paper. Risk measurement evaluates potential losses and is useful for decision making under probabilistic uncertainty. Broadly speaking, fuzzy logic is a form of reasoning based on the 'degree of truth' rather than on the binary true–false principle. But risk measurement and fuzzy systems share a common core theoretical background. Both fields are related to the human behavior under risk, ambiguity or uncertainty.[1] The study of this relationship is a topic of ongoing research from both fields. Goovaerts et al. (2010a), for instance, discuss the hierarchical order between risk measures and decision principles, while Aliev et al. (2012) propose a decision theory under imperfect information from the perspective of fuzzy systems.

Previous attempts to link risk management and fuzzy logic approaches are mainly found in the literature on fuzzy systems. Most authors have focused on the application of fuzzy criteria to financial decision making (Engemann et al., 1996; Gil-Lafuente, 2005; Merigó and Casanovas, 2011), and some have smoothed financial series under fuzzy logic for prediction purposes (Yager and Filev, 1999; Yager, 2008). In the literature on risk management, contributions made by Shapiro (2002, 2004, 2009) regarding the application of fuzzy logic in the insurance context must be remarked.

In this paper we analyze the mathematical relationship between risk measurement and aggregation in fuzzy systems for discrete random variables. A risk measure quantifies the complexity of a random loss in one value that reflects the amount at risk. A key concept in fuzzy systems applications is the aggregation operator, which also allows to combine data into a single value. We show the relationship between the well-known distortion risk measures introduced by Wang (1996) and two specific aggregation operators, the Ordered Weighted Averaging (OWA) operator introduced by Yager (1988) and the Weighted Ordered Weighted Averaging (WOWA) operator introduced by Torra (1997).

Distortion risk measures, OWA and WOWA operators can be analyzed using the theory of measure. Classical measure functions are additive, and linked to the Lebesgue integral. When the

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[1] The expected utility theory by von Neumann and Morgenstern (1947) was one of the first attempts to provide a theoretical foundation to human behavior in decision-making, mainly based on setting up axiomatic preference relations of the decision maker. Similar theoretical approaches are, for instance, the certainty-equivalence theory (Handa, 1977), the cumulative prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992), the rank-dependent utility theory (Quiggin, 1982), the dual theory of choice under risk (Yaari, 1987) and the expected utility without sub-additivity (Schmeidler, 1989), where the respective axioms reflect possible human behaviors or preference relations in decision-making.

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additivity is relaxed, alternative measure functions and, hence, associated integrals are derived. This is the case of non-additive measure functions,[2] often called capacities as it was the name coined by Choquet (1954). We show that the link between distortion risk measures and OWA and WOWA operators is derived by means of the integral linked to capacities, i.e. the Choquet integral. We present the concept of degree of orness for distortion risk measures and illustrate its usefulness.

Our presentation is organized as follows. In Section 2, risk measurement and fuzzy systems concepts are introduced. The relationship between distortion risk measures and aggregation operators is provided in Section 3. An application with some classical risk measures is given in Section 4. Finally, implications derived from these results are discussed in the conclusions.

2. Background and notation

In order to keep this article self-contained and to present the connection between two apparently distant theories, we need to introduce the notation and some basic definitions.

2.1. Distortion risk measures

Two main groups of axiom-based risk measures are *coherent risk measures*, as stated by Artzner et al. (1999), and *distortion risk measures*, as introduced by Wang (1996) and Wang et al. (1997). Concavity of the distortion function is the key element to define risk measures that belong to both groups (Wang and Dhaene, 1998). Suggestions on new desirable properties for distortion risk measures are proposed in Balbás et al. (2009), while generalizations of this kind of risk measures can be found, among others, in Hürlimann (2006) and Wu and Zhou (2006). As shown in Goovaerts et al. (2012), it is possible to link distortion risk measures with other interesting families of risk measures developed in the literature.

The axiomatic setting for risk measures has extensively been developed since seminal papers on coherent risk measures and distortion risk measures. Each set of axioms for risk measures corresponds to a particular behavior of decision makers under risk, as it has been shown, for instance, in Bleichrodt and Eeckhoudt (2006) and Denuit et al. (2006). Most often, articles on axiom-based risk measurement present the link to a theoretical foundation of human behavior explicitly. For example, Wang (1996) shows the connection between distortion risk measures and Yaari's dual theory of choice under risk; Goovaerts et al. (2010b) investigate the additivity of risk measures in Quiggin's rank-dependent utility theory; and Kaluszka and Krzeszowiec (2012) introduce the generalized Choquet integral premium principle and relate it to Kahneman and Tversky's cumulative prospect theory.

Basic risk concepts are formally defined below. Let us set up the notation.

Definition 2.1 (*Probability Space*)**.** A probability space is defined by three elements $(\Omega, \mathcal{A}, \mathcal{P})$. The sample space Ω is a set of the possible events of a random experiment, $\mathcal{A}$ is a family of the set of all subsets of Ω (denoted as $\mathcal{A} \in \wp(\Omega)$) with a σ-algebra structure, and the probability $\mathcal{P}$ is a mapping from $\mathcal{A}$ to $[0, 1]$ such that $\mathcal{P}(\Omega) = 1$, $\mathcal{P}(\emptyset) = 0$ and $\mathcal{P}$ satisfies the σ-*additivity* property.

A probability space is finite if the sample space is finite, i.e. $\Omega = \{\varpi_1, \varpi_2, \ldots, \varpi_n\}$. Then $\wp(\Omega)$ is the σ-*algebra*, which is denoted as 2^{Ω}. In the rest of the article, N instead of Ω will be used when referring to finite probability spaces. Hence, the notation will be $(N, 2^N, \mathcal{P})$.

[2] See Denneberg (1994).

Table 2.1
Correspondence between risk measures and distortion functions.

Risk measure	Distortion function $g(x)$
VaR_α	$\psi_\alpha(x) = \left\{\begin{matrix} 0 & \text{if } x \le 1-\alpha \\ 1 & \text{if } x > 1-\alpha \end{matrix}\right\} = \mathbb{1}_{(1-\alpha,1]}(x)$
TVaR_α	$\gamma_\alpha(x) = \left\{\begin{matrix} \dfrac{x}{1-\alpha} & \text{if } x \le 1-\alpha \\ 1 & \text{if } x > 1-\alpha \end{matrix}\right\} = \min\left\{\frac{x}{1-\alpha}, 1\right\}$

Definition 2.2 (*Random Variable*)**.** Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A random variable X is a mapping from Ω to $\mathbb{R}$ such that $X^{-1}((-\infty, x]) := \{\varpi \in \Omega : X(\varpi) \le x\} \in \mathcal{A},\ \forall x \in \mathbb{R}$.

A random variable X is discrete if $X(\Omega)$ is a finite set or a numerable set without cumulative points.

Definition 2.3 (*Distribution Function of a Random Variable*)**.** Let X be a random variable. The distribution function of X, denoted by F_X, is defined by $F_X(x) := \mathcal{P}\left(X^{-1}((-\infty, x])\right) \equiv \mathcal{P}(X \le x)$.

The distribution function F_X is non-decreasing, right-continuous and $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to+\infty} F_X(x) = 1$. The survival function of X, denoted by S_X, is defined by $S_X(x) := 1 - F_X(x)$, for all $x \in \mathbb{R}$. Note that the domain of the distribution function and the survival function is $\mathbb{R}$ even if X is a discrete random variable. In other words, F_X and S_X are defined for $X(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$ but also for any $x \in \mathbb{R}$.

Definition 2.4 (*Risk Measure*)**.** Let Γ be the set of all random variables defined for a given probability space $(\Omega, \mathcal{A}, \mathcal{P})$. A risk measure is a mapping ρ from Γ to $\mathbb{R}$, so $\rho(X)$ is a real value for each $X \in \Gamma$.

Definition 2.5 (*Distortion Risk Measure*)**.** Let $g : [0, 1] \to [0, 1]$ be a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ (we will call g a distortion function). A distortion risk measure associated to distortion function g is defined by

$$\rho_g(X) := -\int_{-\infty}^{0} [1 - g(S_X(x))]\, dx + \int_{0}^{+\infty} g(S_X(x))\, dx.$$

The simplest distortion risk measure is the mathematical expectation, which is obtained when the distortion function is the identity as shown in Denuit et al. (2005). The two most widely used distortion risk measures are the Value-at-Risk (VaR_α) and the Tail Value-at-Risk (TVaR_α), which depend on a parameter $\alpha \in (0, 1)$ usually called the confidence level. Broadly speaking, the VaR_α corresponds to a percentile of the distribution function. The TVaR_α is the expected value beyond this percentile[3] if the random variable is continuous. The former pursues to estimate what is the maximum loss that can be suffered with a certain confidence level. The latter evaluates what is the expected loss if the loss is larger than the VaR_α. Both risk measures are distortion risk measures with associated distortion functions shown in Table 2.1. Unlike the VaR_α, the distortion function associated to the TVaR_α is concave and, then, the TVaR_α is a *coherent* risk measure in the sense of Artzner et al. (1999). Basically, this means that TVaR_α is subadditive (Acerbi and Tasche, 2002) while the VaR_α is not. Like in the case of VaR_α and TVaR_α, there is a strong relationship between the quantiles of the random variable and distortion risk measures, as it is shown in Dhaene et al. (2012).

[3] We consider TVaR_α as defined in Denuit et al. (2005). That is, $\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\delta(X) d\delta$.