Insurance: Mathematics and Economics 51 (2012) 265–270

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Convex order and comonotonic conditional mean risk sharing

Michel Denuit[a,*], Jan Dhaene[b]

[a] Institut de Statistique, Biostatistique et Sciences Actuarielles - ISBA, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

[b] Actuarial Research Group, AFI, Faculty of Business and Economics, Katholieke Universiteit Leuven, B-3000 Leuven, Belgium

ARTICLE INFO

Article history:
Received September 2010
Received in revised form
April 2012
Accepted 19 April 2012

Keywords:
Stochastic orders
Pareto-optimality
Conditional expectation
Risk sharing
Comonotonicity

ABSTRACT

Using a standard reduction argument based on conditional expectations, this paper argues that risk sharing is always beneficial (with respect to convex order or second degree stochastic dominance) provided the risk-averse agents share the total losses appropriately (whatever the distribution of the losses, their correlation structure and individual degrees of risk aversion). Specifically, all agents hand their individual losses over to a pool and each of them is liable for the conditional expectation of his own loss given the total loss of the pool. We call this risk sharing mechanism the conditional mean risk sharing. If all the conditional expectations involved are non-decreasing functions of the total loss then the conditional mean risk sharing is shown to be Pareto-optimal. Explicit expressions for the individual contributions to the pool are derived in some special cases of interest: independent and identically distributed losses, comonotonic losses, and mutually exclusive losses. In particular, conditions under which this payment rule leads to a comonotonic risk sharing are examined.

1. Introduction and motivation

Loss sharing mechanisms have been studied for decades in the economics and actuarial literatures. The pioneering work by Borch (1960, 1962) considered equilibrium in a reinsurance market. Under appropriate conditions (including that agents are expected utility maximizers and have the same probability on the state space), this author established that any Pareto-optimal loss sharing mechanism is equivalent to a pool arrangement, i.e. all the agents hand their individual losses over to a pool and agree on some rule as to how the total pooled loss has to be divided amongst agents. This fundamental result explains why comonotonicity plays a central role in the study of Pareto-optimality of risk sharing mechanisms, as each component of a comonotonic random vector is (almost surely) equal to a non-decreasing function of the sum of all of its components.

After Borch (1962) established that agents' optimal risk sharing depends only on aggregate loss, Landsberger and Meilijson (1994) have shown that Pareto-optima are comonotonic if agents' preferences agree with second degree stochastic dominance. Specifically, Landsberger and Meilijson (1994) provided an algorithm to construct an improvement of any non-comonotonic risk allocation in the discrete case. This result has been extended to the general case by Dana and Meilijson (2003) and Ludkovski and Rüschendorf (2008). In this paper, we consider the particular conditional mean risk sharing rule and we investigate its comonotonicity and Pareto-optimality. More precisely, we show that whatever the risks faced by decision-makers, there is always a mutually beneficial risk pooling mechanism with respect to second degree stochastic dominance. A noteworthy feature of the analysis conducted in this paper is that risk sharing remains mutually beneficial even if the loss random variables are (positively) correlated. This result is obtained by a standard reduction argument involving conditional expectations, that can be found, e.g., in Dana and Meilijson (2003). In some special cases, explicit expressions for the individual contributions to the pool are derived. We study several particular cases where the risk sharing based on conditional expectations leads to a comonotonic allocation. We also further stress the importance of comonotonicity in the context of Pareto-optimal risk sharing schemes.

Let us briefly describe the contents of this paper. In Section 2, the definition of the convex order is recalled, and some of its basic properties are presented. Section 3 introduces risk sharing and related notions. In Section 4, we define the conditional mean risk allocation and stress the importance of comonotonicity for establishing Pareto-optimality. It is shown that risk-averse decision-makers can always reduce their respective risks by pooling them together. The result guarantees the existence of a mutually beneficial risk exchange. When comonotonic, that risk exchange turns out to be Pareto-optimal. We study the respective contributions of each participant to the pool and establish conditions under which those participants bringing larger losses have to contribute more to the pool, as should hold for any reasonable risk sharing mechanism. In general, the conditional mean risk sharing rule can only be

* Corresponding author.
E-mail addresses: denuit@stat.ucl.ac.be, michel.denuit@uclouvain.be (M. Denuit), jan.dhaene@econ.kuleuven.ac.be (J. Dhaene).

0167-6687/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2012.04.005

applied if we know the conditional distributions of the individual risks, given the total pooled loss. This requires knowing the joint distribution of the individual risks to be pooled. However, there are situations where a weaker form of knowledge is sufficient to apply our conditional mean risk allocation rule. Examples of such situations are given where conditions under which the proposed risk sharing rule produces comonotonic individual payments are also studied. Some particular cases are examined in Section 4: independent and identically distributed losses, comonotonic losses, mutually exclusive losses, and independent losses with log-concave densities.

Henceforth, all the equalities between random variables and random vectors are assumed to hold almost surely, unless stated otherwise.

2. Convex order

Let X and Y be two random variables such that

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)] \quad \text{for all convex functions } g : \mathbb{R} \to \mathbb{R}, \tag{2.1}$$

provided the expectations exist. Then X is said to be smaller than Y in the convex order (denoted as $X \preceq_{\mathrm{CX}} Y$). Now, X is said to be strictly smaller than Y in convex order, which is denoted as $X \prec_{\mathrm{CX}} Y$, if $X \preceq_{\mathrm{CX}} Y$ holds true and X and Y are not identically distributed.

The stochastic inequality $X \preceq_{\mathrm{CX}} Y$ intuitively means that X and Y have the same magnitude (as $\mathbb{E}[X] = \mathbb{E}[Y]$ holds) but that Y is more variable than X. For instance, the variance of Y is larger than the variance of X. For a thorough description of the convex order and its applications in an actuarial context, we refer the reader, e.g., to Denuit et al. (2005).

An important characterization of $\preceq_{\mathrm{CX}}$ is as follows. The random variables X and Y satisfy $X \preceq_{\mathrm{CX}} Y$ if, and only if, there exist two random variables $\widetilde{X}$ and $\widetilde{Y}$, defined on the same probability space, such that $\widetilde{X}$ and X (resp. $\widetilde{Y}$ and Y) are identically distributed, and

$$\mathbb{E}[\widetilde{Y} | \widetilde{X}] = \widetilde{X}. \tag{2.2}$$

More generally, whatever the random variable (or random vector) Z,

$$\mathbb{E}[X|Z] \preceq_{\mathrm{CX}} X. \tag{2.3}$$

The economic intuition behind (2.3) is that averaging a loss (i.e., taking a conditional expectation of it) decreases the risk involved (in the sense of convex order). Applications of (2.2)–(2.3) to actuarial science are described in Denuit and Vermandele (1998, 1999). See also Leitner (2004, 2005) for a use of (2.2) in connection with risk measures and Dhaene et al. (2002a,b) for an application of (2.3) in connection with (comonotonic) approximations for sums of non-independent random variables.

The convex order can also be characterized by means of Tail-VaR risk measures. Recall that the Value-at-Risk (or VaR) for a risk X with distribution functions F_X is defined as

$$\mathrm{VaR}[X; p] = F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \quad 0 < p < 1.$$

The Tail-VaR at probability level p is then defined as

$$\mathrm{TVaR}[X; p] = \frac{1}{1-p} \int_p^1 \mathrm{VaR}[X; \epsilon]\, d\epsilon.$$

Then, $X \preceq_{\mathrm{CX}} Y$ if, and only if, $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathrm{TVaR}[X; p] \leq \mathrm{TVaR}[Y; p]$ holds for all p. See, e.g., Denuit et al. (2005). We will use this characterization of convex order in the proof of our main result. Notice that $X \prec_{\mathrm{CX}} Y$ implies that there exists a probability level $p_0 \in (0, 1)$ such that $\mathrm{TVaR}[X; p_0] < \mathrm{TVaR}[Y; p_0]$.

3. Risk sharing

3.1. Definitions

Consider n decision-makers (economic agents), numbered $i = 1, 2, \ldots, n$. Each of them faces a possible risk (or loss), denoted by X_i. No particular assumption is made about the distribution of the random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$.

Definition 3.1 (*Risk Sharing Scheme*)**.** Consider a portfolio of risks represented by the random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. A risk sharing (or risk allocation) scheme for $\boldsymbol{X}$ is a random vector $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X}))$ where the (measurable) functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are such that

$$\sum_{i=1}^n h_i(\boldsymbol{X}) = \sum_{i=1}^n X_i. \tag{3.1}$$

In the end, each agent will pay $(h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \ldots, h_n(\boldsymbol{x}))$ where $\boldsymbol{x}$ is the observed realization of $\boldsymbol{X}$. The condition (3.1) is called the full risk allocation condition. Consider n economic agents facing total risk

$$S = \sum_{i=1}^n X_i. \tag{3.2}$$

In the sequel we will exclusively use the notation S for the total risk (3.2) of the portfolio $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. The risk sharing scheme characterized by $(h_1, h_2, \ldots, h_n)$ allocates the total risk S to the different agents. The i-th agent bears the risk $h_i(\boldsymbol{X})$, $i = 1, 2, \ldots, n$. Notice that we allow the h_i to be depending on (the distribution of) $\boldsymbol{X}$, as it will be the case for the conditional mean risk allocation discussed in the next section.

An important subclass of risk allocations consists of

$$(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X})) = (g_1(S), g_2(S), \ldots, g_n(S))$$

for some functions $g_1, g_2, \ldots, g_n : \mathbb{R} \to \mathbb{R}$. We will call a risk allocation scheme fulfilling this property a risk pooling scheme.

3.2. Pareto-optimality

In this paper, we study Pareto optimal risk sharing schemes. The following definition is in line with Dana and Meilijson (2003).

Definition 3.2 (*Pareto Optimal Risk Sharing Schemes*)**.** A risk sharing scheme $(h_1^\star(\boldsymbol{X}), h_2^\star(\boldsymbol{X}), \ldots, h_n^\star(\boldsymbol{X}))$ for $\boldsymbol{X}$ is Pareto-optimal if there exists no risk sharing scheme $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X}))$ for $\boldsymbol{X}$ such that the stochastic inequalities

$$h_i(\boldsymbol{X}) \preceq_{\mathrm{CX}} h_i^\star(\boldsymbol{X})$$

hold for $i = 1, 2, \ldots, n$, with at least one of these convex order inequalities being strict.

Hence, we have that a risk sharing scheme is Pareto-optimal if no agent can be made strictly better off (in the sense of convex order) without worsening the situation of another agent. Notice that we define here better in terms of convex order. In the expected utility paradigm, one has that a risk sharing scheme is Pareto-optimal if there exists no risk sharing scheme that increases the expected utility of all (risk-averse assumed) agents, with a strict increase for at least one of them.

Remark 3.3. Note that the convex order naturally appears in the context of Pareto-optimality, because of the condition (3.1) which