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# Asymptotic consistency and inconsistency of the chain ladder

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ABSTRACT

The distribution-free chain ladder reserving method belongs to the most frequently used approaches in general insurance. It is well known, see Mack (1993), that the estimators $\widehat{f_j}$ of the development factors are unbiased and mutually uncorrelated under some mild conditions on the mean structure and under the assumption of independence of the claims in different accident years. In this article we deal with some asymptotic properties of $\widehat{f_j}$. Necessary and sufficient conditions for asymptotic consistency of the estimators of true development factors $f_j$ are provided. A rate of convergence for the consistency is derived. Possible violation of these conditions and its consequences are discussed, and some practical recommendations are given. Numerical simulations and a real data example are provided as well.



## 1. Introduction

Claims reserving is a classical problem in general insurance. A number of various methods have been invented in this field, see England and Verrall (2002) or Wüthrich and Merz (2008) for an overview. Among them, the chain ladder method is probably the most popular and frequently used one for estimating outstanding claims reserves. Besides its simplicity, this approach leads to reasonable estimates of the outstanding loss liabilities under quite mild assumptions on the mean structure and under the assumption of independence of the observations in different accident years.

In recent years, many authors investigated the relationship between various stochastic models and the chain ladder technique, see for instance Mack (1994b) or Renshaw and Verrall (1998). A number of different properties of the estimated ultimate claims amount have been studied. The distribution-free approach introduced by Mack (1993) is probably the most famous one.

In this article we deal with some *asymptotic properties* of the estimators of development factors within this distribution-free framework. Various types of the *conditional asymptotic consistency* are defined. Necessary and sufficient conditions for being the development factors' estimates conditionally consistent are proved and discussed.

### 1.1. Notation

We introduce the classical claims reserving notation and terminology. Outstanding loss liabilities are structured in so-called claims development triangles. Let us denote $X_{i,j}$ all the claim amounts in development year $j$ with accident year $i$. Therefore, $X_{i,j}$ stands for the *incremental claims* in accident year $i$ made in accounting year $i+j$. The current year is $n$, which corresponds to the most recent accident year and development period as well. That is, our data history consists of right-angled isosceles triangles $X_{i,j}$, where $i = 1, \dots, n$ and $j = 1, \dots, n + 1 - i$.

Suppose that $C_{i,j}$ are *cumulative payments* or *cumulative claims* in origin year $i$ after $j$ development periods, i.e., $C_{i,j} = \sum_{k=1}^{j} X_{i,k}$. Hence, $C_{ij}$ is a random variable of which we have an observation if

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**Table 1**
Run-off triangle for cumulative claims $C_{i,j}$.

| Accident year $i$ | Development year $j$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | $\cdots$ | $n-1$ | $n$ |
| 1 | $C_{1,1}$ | $C_{1,2}$ | $\cdots$ | $C_{1,n-1}$ | $C_{1,n}$ |
| 2 | $C_{2,1}$ | $C_{2,2}$ | $\cdots$ | $C_{2,n-1}$ | |
| | | | $\ddots$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $C_{i,n+1-i}$ | | |
| $n-1$ | $C_{n-1,1}$ | $C_{n-1,2}$ | | | |
| $n$ | $C_{n,1}$ | | | | |

$i+j < n+1$ (run-off triangle, see Table 1). The aim is to estimate the ultimate claims amount $C_{i,n}$ and the outstanding claims reserve $R_i = C_{i,n} - C_{i,n+1-i}$ for all $i = 2, \ldots, n$.

## 1.2. Distribution-free approach

The *distribution-free chain ladder* reserving technique is still one of the most frequently used approaches in non-life reserving.

Suppose that $\{C_{i,j}\}_{i,j=1}^{n,n}$ are random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Assume the following stochastic assumptions:

(1) $\mathsf{E}[C_{i,j+1}|C_{i,1}, \ldots, C_{i,j}] = f_j C_{i,j}$, $1 \le i \le n$, $1 \le j \le n-1$;
(2) $\mathsf{Var}[C_{i,j+1}|C_{i,1}, \ldots, C_{i,j}] = \sigma_j^2 C_{i,j}$, $1 \le i \le n$, $1 \le j \le n-1$;
(3) accident years $[C_{i,1}, \ldots, C_{i,n}]$, $1 \le i \le n$ are independent vectors.

These stochastic assumptions correspond to the distribution-free approach of Mack (1993). The parameters $f_j$ are referred to as *development factors*. If $n$ years of the claims history are available then the *estimates of the development factors* based on the chain ladder method are given as

$$\widehat{f}_j^{(n)} = \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}}, \quad 1 \le j \le n-1; \tag{1}$$

$$\widehat{f}_n^{(n)} \equiv 1 \quad \text{(assuming no tail)}.$$

The upper index in (1) is used in order to emphasize that the estimate of development factor $f_j$ depends on $n$ years of history, i.e., we prefer $\widehat{f}_j^{(n)}$ more than $\widehat{f}_j$ from the formal point of view. The ultimate claims amounts $C_{in}$ are estimated by

$$\widehat{C}_{in} = C_{i,n+1-i} \times \widehat{f}_{n+1-i}^{(n)} \times \cdots \times \widehat{f}_{n-1}^{(n)}.$$

Mack (1993) proved that the estimators $\widehat{f}_j^{(n)}$ are unbiased and mutually uncorrelated under assumptions (1) and (3) together with an additional assumption

(4) $\sum_{i=1}^{n-j} C_{i,j} > 0, \quad 1 \le j \le n-1.$

Furthermore, assumption (2) is essential for the calculation of the mean squared error and the standard error of $\widehat{C}_{in}$. It has to be remarked that assumption (2) straightforwardly postulates a condition that $C_{i,j} \ge 0$ [P]-a.s. for all $i, j \in \mathbb{N}$.

If the cumulative claim $C_{i,n+1-i}$ is equal to zero for some particular accident year $i \in \{1, \ldots, n\}$, then all the consequent predictions $\widehat{C}_{i,j}$, where $j > n+1-i$, are zeros as well. But this situation occurs very exceptionally. Independence assumption (3) can sometimes be viewed as slightly unrealistic. In these cases, the chain ladder does not seem to be a suitable choice in reserving. Nevertheless, the assumptions of a distribution-free chain ladder were thoroughly discussed many times and we refer the reader for completeness to relevant articles, e.g., Mack (1994a) or Mack (1994b).

## 1.3. Properties of development factors' estimators

*Unbiasedness* of the development factors estimators $\widehat{f}_j^{(n)}$ is often stressed out as an important advantageous property of the chain ladder method. However, unbiasedness of an estimator as such is from the statistical point of view of less importance compared to the consistency. A simple example illustrates why. Suppose that $Y_1, \ldots, Y_n$ are iid variables sampled from a distribution with finite mean $\mathsf{E}Y$. One can use $T_1(Y_1, \ldots, Y_n) = Y_1$ as an estimator of the unknown mean $\mathsf{E}Y$. This would be, of course, very naive in practice as only the first observation from the sample is used and directly taken as the estimator. However, we take this example because of its simplicity. The estimator $T_1$ is obviously unbiased, but surely inconsistent. On the other hand, the estimator $T_2(Y_1, \ldots, Y_n) = \frac{1}{n}\sum_{i=1}^n Y_i + \frac{1}{n}$ is biased, but *consistent*. It is easy to see that $T_2$ approaches $\mathsf{E}Y$ with probability one as $n$ tends to infinity. Alternatively speaking, for $n$ large enough, $T_2$ is very close to the sample average and, hence, provides a reasonable estimate for the mean.

The latter simple example illustrates that the unbiasedness of the development factors' estimators does not guarantee reasonable estimates of the ultimate claims. From an actuarial point of view, the consistency of $\widehat{f}_j^{(n)}$ might be a more tempting property of the method. It ensures that a sufficiently large number of observations leads to estimates close to the true quantity. In the claims reserving problem this means that the method provides accurate estimates of the outstanding loss of liabilities.

However, the consistency of an estimator is an important property for other reasons as well. Let us present one of them. If $\widehat{f}_j^{(n)}$ is an unbiased estimator of $f_j$, then this does not imply (and in the majority of cases it is not true) that $[\widehat{f}_j^{(n)}]^{-1}$ is an unbiased estimate of $f_j^{-1}$. In general, $[\widehat{f}_j^{(n)}]^{-1}$ can behave quite unpredictably. On the other hand, if $\widehat{f}_j^{(n)}$ is a consistent estimator of $f_j$, then $[\widehat{f}_j^{(n)}]^{-1}$ is a consistent estimator of $f_j^{-1}$. In general, a *continuous transformation* preserves the property of being a consistent estimator (continuous mapping theorem). This is very useful in many applications. For instance, consider the Bornhuetter–Ferguson method (BF), see Wüthrich and Merz (2008), for reserves estimates. The claims development pattern $\beta_j$ is sometimes estimated using $\widehat{f}_j^{(n)}$ as

$$\widehat{\beta}_j^{(n)} = \prod_{k=j}^{n-1} \frac{1}{\widehat{f}_k^{(n)}}.$$

Hence, the consistency of $\widehat{f}_j^{(n)}$ implies the consistency of $\widehat{\beta}_j^{(n)}$. On the other hand, the unbiasedness of $\widehat{f}_j^{(n)}$ does not "transfer" to $\widehat{\beta}_j^{(n)}$ in any sense.

Furthermore, an estimate of the *mean squared error (MSE) of reserves* depends on the estimates of development factors and the dependence is not linear (Mack, 1993, Theorem 3). The same holds for the *MSE of prediction*. Therefore, the unbiasedness of $\widehat{f}_j^{(n)}$ does not preserve the unbiasedness for estimates of the MSE of reserves or prediction. Contrary to unbiasedness, the consistency of development factors' estimates $\widehat{f}_j^{(n)}$ also *guarantees* the consistency of the reserves' or prediction's MSE.

Finally, the estimator not only has to stay on target asymptotically but its *variability* (usually measured by variance) also has to shrink, leading to better accuracy. Since the consistency is only a qualitative property of the estimate, it is needed to characterize the consistency of the development factors' estimates from a quantitative point of view. Indeed, the variance of the estimates will provide us a *rate of convergence* of the estimates, as will be pointed out later.