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Stationary-excess operator and convex stochastic orders

Claude Lefèvre [a], Stéphane Loisel [b,*]

[a] Département de Mathématique, Université Libre de Bruxelles, Boulevard du Triomphe C.P.210, B-1050 Bruxelles, Belgique

[b] Université de Lyon, Université Claude Bernard Lyon 1, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France

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ABSTRACT

The present paper aims to point out how the stationary-excess operator and its iterates transform s-convex stochastic orders and the associated moment spaces. This allows us to propose a new unified method on constructing s-convex extrema for distributions that are known to be t-monotone. Both discrete and continuous cases are investigated. Several extremal distributions under monotonicity conditions are derived. They are illustrated with some applications in insurance.

1. Introduction

Concepts of stochastic orderings are useful in a number of applied probability models. This is especially true in insurance and finance when different risk scenarios are possible and have to be compared. Such issues arise, for instance, in life insurance, ruin theory and portfolio analysis. A theory of stochastic orders with various applications can be found in Goovaerts et al. (1990), Kaas et al. (1994), Ross (1996), Müller and Stoyan (2002) and Shaked and Shanthikumar (1994), (2007).

This paper is concerned with the class of *s-convex* stochastic orders, where $s \in \mathbb{N}_0 = \{1, 2, \ldots\}$. For $s = 1$, this order is a classical stochastic dominance. For $s = 2$, it is a well-known convex order and corresponds, in actuarial sciences, to a stop-loss order with fixed mean. For an arbitrary s, the s-convex order compares the sth right-tail distribution functions of random variables that have the same first $s - 1$ moments. In the case of discrete distributions, this class of orders was studied in Lefèvre and Picard (1993), Fishburn and Lavalle (1995), Lefèvre and Utev (1996), Denuit and Lefèvre (1997) and Denuit et al. (1999a,c), among others. The more traditional case of real-valued random variables was investigated by Rolski (1976), Levy (1992) and Denuit et al., (1998, 1999b) and in many other works.

A directly related question is the construction of stochastic extrema with respect to s-convex orders. In an actuarial context, the extremal risks represent the less and more dangerous risks, and their knowledge can yield accurate lower and upper bounds on various risk quantities of interest (the premium, for instance). The problem of s-convex optimization corresponds to a traditional moment problem; see, e.g. Hoeffding (1955), Karlin and Studden (1966), Kemperman (1968), Utev (1985), Prékopa (1990) and Hürlimann (1999). In the discrete case, explicit extremal distributions are known for $s = 1, 2, 3, 4$ and were derived by Denuit and Lefèvre (1997), Denuit et al. (1999c) and Courtois et al. (2006). In the real case, explicit extrema were obtained by Jansen et al. (1986), De Vylder (1996) and Denuit et al. (1998, 1999b), inter alia.

The *stationary-excess* operator is a standard mathematical tool that plays an important role in renewal theory and survival analysis (Cox, 1972). Our purpose in the present work is to investigate how this operator and its iterates transform s-convex stochastic orders and the associated moment spaces. Both discrete and continuous cases are discussed, with a special emphasis on the less traditional discrete case.

As a corollary, the following approach allows us to propose a method on solving the s-convex optimization problem within the subset of distributions that are known to be *t-monotone*, $t \in \mathbb{N}_0$.

* Corresponding author.
E-mail address: stephane.loisel@univ-lyon1.fr (S. Loisel).

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For $t = 1$, this property is the nonincreasingness of the distributions, and for $t = 2$, it corresponds to their nonincreasingness and convexity. Monotonicity of order t in the discrete (resp. continuous) case means that the first t differences of the probability mass function (resp. derivatives of the density function) are assumed to be of alternating signs with a negative sign at the beginning. Our key theorem states that a problem of s-convex optimization among t-monotone distributions is (almost) equivalent, through the stationary-excess operator, to a problem of $s+t$-convex optimization without monotonicity constraints.

It was already pointed out in several papers that s-convex extrema can be improved when the distributions of interest are nonincreasing (and, more generally, unimodal); see, e.g., Denuit et al. (1998, 1999b, 1999c). Our work can be viewed as an extension of these results that is carried out by using the stationary-excess operator and its iterates.

This paper is organized as follows. In Section 2, we present the stationary-excess operator, under its usual definition and in a non-standard version specific to discrete distributions. In Section 3, we prove that this operator essentially transforms any $s+1$-convex stochastic order to an s-convex stochastic order for nonincreasing distributions. The result is extended in Section 4 by showing that the tth iterate of the operator transforms any $s+t$-convex order to an s-convex order for t-monotone distributions. In Section 5, we use this property, together with well-known convex extrema (without monotonicity conditions), to derive several explicit convex extrema for nonincreasing, possibly convex, distributions. Finally, the interest of these extrema is illustrated in Section 6 with some applications to insurance.

This work has been presented at the "(2b) or not (2b) Conference" organized in June 2009 at the Université of Lausanne, in honor to Professor Hans U. Gerber. It gives us an opportunity to point out a nice paper by Keilson and Gerber (1971) *on a related notion of discrete unimodality. We also thank Professor F. Dufresne for the excellent organization of the Conference.*

2. Stationary-excess operator

In its classical version, the stationary-excess operator, H for example, is built for any non-negative random variable X with distribution function F_X and mean $\mathrm{E}(X) > 0$ (see Definition 2.3). It is worth recalling that in a renewal process, if an interval between points is of distribution function F_X, then the associated stationary-excess mapping $\mathrm{H}(F_X)$ gives the distribution function of the interval to the next point from an arbitrary time in equilibrium. To begin with, we are going to introduce a similar operator H that is specific to discrete distributions.

2.1. Discrete version

Let us assume that X is a *discrete non-negative* random variable with probability mass function $P_X = \{P(X = j), j \in \mathbb{N} = \{0, 1, \ldots\}\}$ and mean $\mathrm{E}(X) > 0$. Obviously, the classical stationary-excess operator may be applied here too. As explained by Whitt (1985), however, it is more appropriate to work with a discrete version that is directly applicable to discrete-time renewal processes. We propose to adopt the following definition.

Definition 2.1. A discrete stationary-excess operator H maps any such random variable X to an associated discrete non-negative random variable X_H whose probability mass function $\mathrm{H}(P_X)$ is defined by

$$\mathrm{H}(P_X)(j) \equiv P(X_\mathrm{H} = j) = \frac{P(X \geq j+1)}{\mathrm{E}(X)}, \quad j \in \mathbb{N}. \tag{2.1}$$

Let us notice that a slightly different operator was investigated by Whitt (1985) for a discrete positive random variable X. Specifically, the associated random variable X_H is of probability mass function defined by

$$P(X_\mathrm{H} = j) = \frac{P(X \geq j)}{\mathrm{E}(X)}, \quad j \in \mathbb{N}_0.$$

Contrary to that operator, H defined by (2.1) does not yield a one-to-one correspondence on the set of probability measures on $\mathbb{N}$. Indeed, it is directly checked that X_H and $(\nu X)_\mathrm{H}$ are equidistributed if ν is an indicator independent of X. Nevertheless, when $\mathrm{E}(X)$ is fixed, H gives a one-to-one correspondence since

$$P(X = 0) = 1 - \mathrm{E}(X) P(X_\mathrm{H} = 0), \quad \text{and} \tag{2.2}$$

$$P(X = j) = \mathrm{E}(X)[P(X_\mathrm{H} = j-1) - P(X_\mathrm{H} = j)], \quad j = 1, 2, \ldots. \tag{2.3}$$

The definition (2.1) has the advantage of leading to a simple relationship between the binomial moments of X_H and X. We make the convention $\binom{x}{y} = 0$ if $x < y$.

Lemma 2.2.

$$\mathrm{E}\binom{X_\mathrm{H}}{i} = \frac{1}{\mathrm{E}(X)} \mathrm{E}\binom{X}{i+1}, \quad i \in \mathbb{N}. \tag{2.4}$$

Proof. Let us consider the iterated right-tail distribution functions of X, i.e. $\bar{F}_0(X, j) = P(X = j)$ and

$$\bar{F}_{i+1}(X, j) = \sum_{k=j}^{\infty} \bar{F}_i(X, k), \quad i, j \in \mathbb{N}.$$

As proved in Lefèvre and Utev (1996), an equivalent expression is

$$\bar{F}_{i+1}(X, j) = \mathrm{E}\binom{X - j + i}{i}, \quad i, j \in \mathbb{N}. \tag{2.5}$$

Let us turn to the iterated right-tail distribution functions of X_H. By (2.1), we have

$$\bar{F}_0(X_\mathrm{H}, j) = P(X_\mathrm{H} = j) = \frac{\bar{F}_1(X, j+1)}{\mathrm{E}(X)}, \quad j \in \mathbb{N}.$$

Arguing by induction, we then find

$$\bar{F}_{i+1}(X_\mathrm{H}, j) = \frac{\bar{F}_{i+2}(X, j+1)}{\mathrm{E}(X)}, \quad i, j \in \mathbb{N}. \tag{2.6}$$

Finally, taking $j = i$ in (2.5) and (2.6), we obtain the identity (2.4). □

2.2. Continuous version

Let X be any *non-negative* real random variable with distribution function F_X and mean $\mathrm{E}(X) > 0$. The usual stationary-excess operator is defined as follows (Cox, 1972). For ease, it will be named continuous subsequently.

Definition 2.3. A continuous stationary-excess operator H maps any such random variable X to an associated non-negative random variable X_H whose distribution function $\mathrm{H}(F_X)$ is defined by

$$\mathrm{H}(F_X)(x) \equiv P(X_\mathrm{H} \leq x) = \frac{1}{\mathrm{E}(X)} \int_0^x [1 - F_X(y)] \mathrm{d}y, \quad x \geq 0.$$