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Optimality of general reinsurance contracts under CTE risk measure

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ABSTRACT

By formulating a constrained optimization model, we address the problem of optimal reinsurance design using the criterion of minimizing the conditional tail expectation (CTE) risk measure of the insurer's total risk. For completeness, we analyze the optimal reinsurance model under both binding and unbinding reinsurance premium constraints. By resorting to the Lagrangian approach based on the concept of directional derivative, explicit and analytical optimal solutions are obtained in each case under some mild conditions. We show that pure stop-loss ceded loss function is always optimal. More interestingly, we demonstrate that ceded loss functions, that are not always non-decreasing, could be optimal. We also show that, in some cases, it is optimal to exhaust the entire reinsurance premium budget to determine the optimal reinsurance, while in other cases, it is rational to spend less than the prescribed reinsurance premium budget.

1. Introduction

Since the seminal papers by Borch (1960) and Kahn (1961), the quest for optimal reinsurance has remained a fascinating area of research and it has drawn significant interest from both academicians and practitioners. Numerous creative models have been proposed with elegant mathematical tools, and sophisticated optimization theories have also been used in deriving the optimal solutions to the proposed models. The fascination with the optimality of reinsurance stems from its potential as an effective risk management tool for insurers. Indeed, by resorting to a meticulous choice of reinsurance treaty, it allows the insurer to control better and thereby manage its risk exposure. The use of reinsurance, on the other hand, incurs an additional cost to the insurer in the form of reinsurance premium. Naturally, the larger the expected risk that is transferred to a reinsurer, the higher the reinsurance premium. This implies that an insurer has to deal with the classical risk and reward tradeoff in balancing the amount of risk retained and risk transferred.

In this paper, we assume a single-period setting. The optimal reinsurance treaty is typically determined by solving an optimization problem, which could involve either maximization or minimization, depending on the chosen criterion. For example, one of the most classical results is based on the variance minimization model. It states that pure stop-loss reinsurance is the optimal treaty in the sense that it yields the least variance of the insurer's retained loss among all the treaties with the same pure premium; see, for example Kaas et al. (2001). Another classical result corresponds to the utility maximization model, which is attributed to Arrow (1974). It asserts that stop-loss reinsurance maximizes the expected utility of the insurer, provided that the insurer has a concave utility function.

In recent years, extensive research on optimal reinsurance has been conducted by Kaluszka (2001, 2004a,b, 2005), who derived explicit optimal reinsurance policies on a number of ingenious risk measure based reinsurance models. Other related contributions include Gajek and Zagrodny (2000, 2004), Promislow and Young (2005), Balbás et al. (2009) and the references therein. Recent relevant papers on the expected utility maximization models include Zhou and Wu (2008) and Zhou et al. (2010).[1]

More recently, two important risk measures known as the Value-at-Risk (VaR) and the conditional tail expectation (CTE) have

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[1] Both papers analyze the optimal insurance purchase, but their results could be applied to the optimal reinsurance purchase.

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been applied to insurance and reinsurance for the determination of optimal policies. This area of research is inspired by the prevalent use of these two risk measures among banks and insurance companies for risk assessment and for determining regulatory capital requirement (see, for example, Wang et al. (2005), Huang (2006), Cai and Tan (2007), Cai et al. (2008), Bernard and Tian (2009), Balbás et al. (2009), and Tan and Weng (2010)). In particular, Bernard and Tian (2009) analyzed the optimal reinsurance contracts under two tail risk measures: a VaR-like risk measure (the probability for the underlying loss to exceed a given threshold) and a CTE-like risk measure (the expected loss over a given threshold); see Remark 2.2 for more detailed comments. Cai and Tan (2007), Cai et al. (2008) and Tan and Weng (2010) derived the optimal reinsurance treaties under the strict definition of VaR and CTE. While the optimal reinsurance obtained in these three papers are explicit, one critical limitation is the lack of generality in that the optimality of the reinsurance designs is confined to reinsurance treaties of specific structure. For example, Cai and Tan (2007) assumed that the feasible ceded loss function is of the form stop-loss, while Cai et al. (2008) and Tan and Weng (2010) restricted to the class of increasing convex functions. Balbás et al. (2009) characterized the optimal reinsurance treaties under a very general risk measure including CTE as one of the special cases.

The objective of the paper is to explicitly derive the optimal solutions over all possible reinsurance treaties using the criterion of minimizing CTE of the insurer's resulting risk. Because of the generality of the optimal reinsurance model, we will see shortly that this is a mathematically more complex problem. In fact, our formulation of the reinsurance model entails us in solving some convex optimization problem over a Hilbert space using the Lagrangian method. Because the objective function is only directionally differentiable but not Gâteaux differentiable, we utilize the concept of directional derivative in searching for the optimal solutions.

It is interesting to note that pure stop-loss reinsurance is always optimal under our CTE minimization model, a result which is consistent with the variance minimization and expected utility maximization reinsurance models. More interestingly, we also establish formally that ceded loss function of other structures (such as those that do not need to be always non-decreasing) could also be optimal. Moreover, it should be emphasized that our proposed reinsurance model is a constrained optimization model in that one of the constraints can be interpreted as either a reinsurance premium budget or an insurer's profitability guarantee. For completeness, we analyze the optimal solutions under both binding and unbinding cases depending on the optimal reinsurance premium expenditure relative to the reinsurance premium budget. Enforcing the reinsurance premium budget constraint to be binding, it facilitates us in establishing the optimal risk and reward profile and hence leads to the insurer's reinsurance efficient frontier. On the other hand, if the reinsurance premium budget constraint does not have to be binding, then there are cases where it is optimal to spend less than the prescribed budget.

The remaining paper is organized as follows. Section 2 gives some preliminaries and describes the setup of the proposed reinsurance models. Section 3 states the solutions to our proposed optimal reinsurance models with an unbinding constraint. Remarks and numerical examples to further elaborate these key results are also provided in the same section. Section 4 discusses the optimal solutions to the binding reinsurance model. Section 5 concludes the paper. Key mathematical background with respect to the optimization theory in Banach spaces, together with some relevant concepts related to the directional derivative are collected in Appendix. The proofs of all the propositions and theorems are also given in the same Appendix.

2. Preliminaries and reinsurance model

Let X denote the (aggregate) loss initially assumed by an insurer. Suppose X is a nonnegative random variable, and identify it by a probability measure Pr on the measurable space $(\Omega, \mathcal{F})$ with $\Omega = [0, \infty)$ and $\mathcal{F}$ being the Borel σ-field on Ω, such that the distribution function F_X of the underlying risk X is defined by $F_X(t) = \Pr\{[0, t]\}$ for $t \geq 0$. It is worth noting that the distribution of the loss random variable defined in such a way it is general enough for modeling a loss distribution. It can be any of a general distribution, not necessarily either continuous or discrete. Denote by $f(X)$ the part of loss transferred from the insurer to a reinsurer in the presence of the reinsurance. The function $f : [0, \infty) \mapsto [0, \infty)$, satisfying $0 \leq f(x) \leq x$ for all $x \geq 0$, is known as the ceded loss function or the indemnification function. Associated with the ceded loss function $f(X)$, we denote $I_f(X) := X - f(X)$ as the retained loss function of the insurer in the presence of reinsurance. Similarly, I_f can also be recognized as a function $I_f : [0, \infty) \mapsto [0, \infty)$. By transferring part of its loss to the reinsurer, the insurer is obligated to pay the reinsurance premium $\Pi(f(X))$ to the reinsurer, where Π is a principle adopted for calculating the reinsurance premium. Consequently, the total cost or the total risk for the insurer in the presence of reinsurance, denoted by $T_f(X)$, is the sum of the retained loss and the reinsurance premium,[2] i.e.,

$$T_f(X) = I_f(X) + \Pi(f(X)) = X - f(X) + \Pi(f(X)). \tag{2.1}$$

In situation where there is no ambiguity on the explicit dependence on the random variable X, we simplify the notation by writing $f(X)$, $I_f(X)$ and $T_f(X)$ as f, I_f and T_f, respectively.

Eq. (2.1) demonstrates clearly the intricate role of the reinsurance treaty f on the resulting total risk T_f. A more conservative insurer could reduce its risk exposure by transferring most of the risk to a reinsurer at the expense of higher reinsurance premium. On the other hand, a more aggressive insurer could reduce the cost of reinsurance by exposing to a greater expected risk. This illustrates the classical tradeoff between risk retained and risk transferred. In determining the optimal reinsurance treaties, one prudent strategy from the insurer's perspective is to minimize the resulting risk exposure $T_f(X)$ in terms of an appropriately chosen risk measure. In this paper, we focus on the risk measure CTE risk measure.

Before providing a formal definition of CTE, it is necessary to define a closely related risk measure known as the Value-at-Risk (VaR):

Definition 2.1. The VaR of a loss random variable Z at a confidence level $1 - \alpha$, $0 < \alpha < 1$, is formally defined as

$$\mathrm{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R} : \Pr(Z \leq z) \geq 1 - \alpha\}. \tag{2.2}$$

In practice, the parameter α typically is a small value such as 5% or even 1%. Consequently, $\mathrm{VaR}_\alpha(Z)$ captures the underlying risk exposure by ensuring that with a high degree of confidence (such as $1 - \alpha$ probability) the loss will not exceed the VaR level. While VaR is intuitive and is widely accepted among financial institutions as a risk measure for market risk, it is often criticized for its inadequacy in capturing the tail behavior of the loss distribution, in addition to its violation of properties such as the subadditivity. To overcome these drawbacks, the risk measure CTE has been proposed. CTE is defined as the expected loss given that the loss falls in the worst α part of the loss distribution.

[2] Alternatively, we can choose to work with the net risk or net loss random variable, $\Gamma(f)$, defined as $\Gamma(f) = T_f - p_0$, where p_0 is the insurance premium collected by the insurer from the policyholders. $\Gamma(f)$ takes into account the insurance premium received by the insurer for underwriting risk X. Because p_0 is a constant, our proposed optimal CTE-based reinsurance models, whether defined via T_f or $\Gamma(f)$, are equivalent due to the translation invariance property of the CTE risk measure.