Insurance: Mathematics and Economics 46 (2010) 281–289

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Conditional law of risk processes given that ruin occurs

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ARTICLE INFO

Article history:
Received July 2009
Received in revised form October 2009
Accepted 22 October 2009

MSC:
60J25
60J75
60J35

Keywords:
Markov process
Generator
Absorbing state
Ruin
Diffusion process
Jump process
Weak convergence
Piecewise deterministic Markov process (PDMP)
Change of measure
Cramér condition
Subexponential distribution

ABSTRACT

A risk process that can be Markovised is conditioned on ruin. We prove that the process remains a Markov process. If the risk process is a PDMP, it is shown that the conditioned process remains a PDMP. For many examples the asymptotics of the parameters in both the light-tailed case and the heavy-tailed case are discussed.

1. Introduction

Consider a risk model $\{X_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ modelled by a piecewise deterministic Markov process (PDMP) on a state space $E \cup \{\Delta\}$; see Davis (1984), Davis (1993), Dassios and Embrechts (1989), Rolski et al. (1999) or Schmidli (1992) for an introduction. We assume that all processes have cadlag paths. The state Δ is absorbing and we call it the state of ruin. We define the time of ruin as

$$\tau = \inf\{t \geq 0 : X_t = \Delta\}$$

and the ruin probability as $\psi(x) = \mathbb{P}_x[\tau < \infty]$. Here $\mathbb{P}_x$ denotes the measure with $\mathbb{P}_x[X_0 = x] = 1$. A main example is the classical risk process, see Section 3, where τ may be defined as $\tau = \inf\{t \geq 0 : X_t < 0\}$. In our modelling $E = [0, \infty)$ and we identify $(-\infty, 0)$ with Δ.

Our main interest in this paper will be to consider the process under the conditional measure

$$\widetilde{\mathbb{P}}[A] = \mathbb{P}[A \mid \tau < \infty].$$

A similar quantity had earlier been considered for special cases by Asmussen (1982), Asmussen and Klüppelberg (1996) and Schmidli (1999a). In these papers weak convergence of the measures $\widetilde{\mathbb{P}}_u$ as $u \to \infty$ is considered. The result is, that in the small claim case – i.e., that enough exponential moments exist – the claim intensity is increased and the claim sizes increase in stochastic order, such that the drift becomes negative. This change is uniform and holds for all claims happening as long the surplus is large enough; i.e., infinite. In the large claim case, intensity and claim sizes asymptotically do not change. Suddenly, a huge claim arrives that will lead to ruin. Whereas the small claim case is understandable – ruin is triggered by more and larger claims than usual – it seems strange that in the large claim case the measures conditioned and not conditioned on ruin asymptotically do not differ until the time of ruin. This is a phenomenon due to weak convergence. The weight of the claims leading to ruin immediately is only changed slightly but the weight put to large values is not negligible for the mean value; see also Theorem 2. Note that, unless ruin occurs almost surely $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent. Indeed, $\widetilde{\mathbb{P}}[\tau < \infty] = 1 > \mathbb{P}[\tau < \infty]$. However, on $\mathcal{F}_t$ the two measures are equivalent with the Radon–Nikodym derivative $\mathrm{d}\widetilde{\mathbb{P}}_u|_{\mathcal{F}_t}/\mathrm{d}\mathbb{P}_u|_{\mathcal{F}_t} = \psi(u)^{-1}\mathbb{E}[\mathbb{1}_{\tau<\infty} \mid \mathcal{F}_t]$.

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0167-6687/$ – see front matter
doi:10.1016/j.insmatheco.2009.10.007

In this paper we want to investigate the process conditioned on ruin for a finite initial capital. We will show that it remains a PDMP under the measure $\widetilde{\mathbb{P}}$ and we will determine the parameters in terms of the ruin probability $\psi(x)$. Note that $\psi(x)$ usually has to be calculated numerically. We will consider several examples of risk processes in this paper. Because no closed-form expression for $\psi(x)$ is known in most cases, we will discuss mainly $\widetilde{\mathbb{P}}$ for large initial capital u, i.e. in the limit as $u \to \infty$. In the Cramér case we find as in Asmussen (1982), that $\widetilde{\mathbb{P}}_u$ tends to the measure normally used to determine the ruin probabilities, see Rolski et al. (1999). For the subexponential case we find as in Asmussen and Klüppelberg (1996) that $\widetilde{\mathbb{P}}_u$ tends to $\mathbb{P}$. For the application in practice, for small initial capital one will have to use numerical results or statistical inference to determine the parameters. If, as in the small claim case, the difference to the unconditioned process is large, estimation will yield a model tending to $-\infty$, and the insurer will change the premium. If the conditioned and unconditioned processes are hard to distinguish, there is no chance to prevent ruin. Since ruin will be caused by a rare event, the surplus process will look well even in the case where the process tends to $-\infty$. In a last section, we show that the results can be applied for jump-diffusion processes as well.

In order to avoid trivialities, we assume that $0 < \psi(x) < 1$ for all $x \in E$. Points with $\psi(x) = 0$ are not interesting and points with $\psi(x) = 1$ can be identified with Δ.

2. PDMPs conditioned on ruin

We start by showing that a Markov process remains a Markov process under the measure $\widetilde{\mathbb{P}}$. The following result holds for any Markov process with absorbing state Δ.

Lemma 1. *Let $\{X_t\}$ be a (strong) Markov process under the measure $\mathbb{P}$. Then $\{X_t\}$ is a (strong) Markov process under the measure $\widetilde{\mathbb{P}}$.*

Proof. Fix $t > 0$ and let B be a Borel set of $E \cup \{\Delta\}$ and $\tilde{B}$ be a Borel set of $(E \cup \{\Delta\})^{[0,t]}$. Then

$$\begin{aligned}
&\widetilde{\mathbb{E}}[\tilde{\mathbb{P}}[X_{t+s} \in B \mid X_t]; \{X_v : 0 \le v \le t\} \in \tilde{B}] \\
&\quad = \frac{\mathbb{E}[\frac{\mathbb{P}[X_{t+s}\in B, \tau<\infty \mid X_t]}{\mathbb{P}[\tau<\infty \mid X_t]}; \{X_v : 0 \le v \le t\} \in \tilde{B}, \tau < \infty]}{\mathbb{P}[\tau < \infty]} \\
&\quad = \frac{\mathbb{E}[\frac{\mathbb{P}[X_{t+s}\in B, \tau<\infty \mid \mathcal{F}_t]}{\mathbb{P}[\tau<\infty \mid \mathcal{F}_t]}; \{X_v : 0 \le v \le t\} \in \tilde{B}, \tau < \infty]}{\mathbb{P}[\tau < \infty]} \\
&\quad = \frac{\mathbb{E}[\frac{\mathbb{P}[X_{t+s}\in B, \{X_v:0\le v\le t\}\in\tilde{B}, \tau<\infty \mid \mathcal{F}_t]}{\mathbb{P}[\tau<\infty \mid \mathcal{F}_t]}; \tau < \infty]}{\mathbb{P}[\tau < \infty]} \\
&\quad = \widetilde{\mathbb{E}}[\tilde{\mathbb{P}}[X_{t+s} \in B, \{X_v : 0 \le v \le t\} \in \tilde{B} \mid \mathcal{F}_t]] \\
&\quad = \tilde{\mathbb{P}}[X_{t+s} \in B, \{X_v : 0 \le v \le t\} \in \tilde{B}].
\end{aligned}$$

This shows that $\tilde{\mathbb{P}}[X_{t+s} \in B \mid \mathcal{F}_t] = \tilde{\mathbb{P}}[X_{t+s} \in B \mid X_t]$. Thus $\{X_t\}$ is a Markov process. If we replace t by a finite stopping time, the strong Markov property can be shown similarly. □

Let us now consider a PDMP $\{X_t\}$ on a subset E of an Euclidean space $\mathbb{R}^d$ with vector-field $\chi(x)$, jump intensity $\lambda(x)$ and jump measure $Q(x, \cdot)$. Note that jumps may be caused by reaching the boundary ∂E of E. The integral curves connected to the vector-field χ, that means the deterministic paths between jumps, are denoted by $\phi(x, t)$ with $\phi(x, 0) = x$. That is, $\frac{\partial}{\partial t} f(\phi(x, t)) = (\chi f)(\phi(x, t))$ for all continuously differentiable functions f.

Let us first discuss the ruin probability. The generator $\mathfrak{A}$ of the PDMP is given by

$$\mathfrak{A}f(x) = \chi f(x) + \lambda(x)\left[\int_{E\cup\{\Delta\}} f(y)Q(x, \mathrm{d}y) - f(x)\right],$$

for all functions f in the domain $\mathcal{D}(\mathfrak{A})$ of the generator. We assume that τ is a stopping time. In particular, this will be the case if the filtration $\{\mathcal{F}_t\}$ is right continuous. We will not discuss these technical problems in this paper.

If the ruin probability $\psi(u)$ is absolutely continuous, then it follows easily that $\mathfrak{A}\psi(x) = 0$ with boundary condition $\psi(\Delta) = 1$. If jumps can be caused by reaching a boundary point, then the boundary condition $\psi(x) = \int_{E\cup\{\Delta\}} \psi(y)Q(x, \mathrm{d}y)$ has to hold for all $x \in \partial^* E$, the set of points $x \in \partial E$ that can be reached from the inside of E within finite time. If the jump measure Q and the intensity have nice properties it is possible to show the above equations directly. There is another possible approach. Suppose there is a bounded function $f(x)$ that is absolutely continuous along integral paths and satisfies $\mathfrak{A}f(x) = 0, f(\Delta) = 0$ and the boundary condition $f(x) = \int_{E\cup\{\Delta\}} f(y)Q(x, \mathrm{d}y)$ for all $x \in \partial^* E$. Then $\{f(X_{\tau\wedge t})\}$ is a local martingale, see Davis (1984). Because f is bounded it is a martingale, indeed. In particular, by the martingale convergence theorem the process has to converge. In most cases considered in the literature the process $\{f(X_t)\}$ will converge to a deterministic constant on $\{\tau = \infty\}$, f_∞ say. Then the martingale stopping theorem gives $\psi(x) = 1 - f(x)/f_\infty$. We, therefore, assume from now on that the ruin probability is in the domain $\mathcal{D}(\mathfrak{A})$ of the generator and therefore satisfies $\mathfrak{A}\psi(x) = 0$.

We now investigate the process under the measure $\widetilde{\mathbb{P}}$, i.e. the measure conditioned on that ruin occurs.

Theorem 1. *Under the measure $\widetilde{\mathbb{P}}$ the process $\{X_t\}$ is a piecewise deterministic Markov process with vector-field χ, jump intensity*

$$\tilde{\lambda}(x) = \frac{\lambda(x)\int_{E\cup\{\Delta\}} \psi(y)Q(x, \mathrm{d}y)}{\psi(x)}$$

and jump measure

$$\tilde{Q}(x, \mathrm{d}y) = \frac{\psi(y)Q(x, \mathrm{d}y)}{\int_{E\cup\{\Delta\}} \psi(z)Q(x, \mathrm{d}z)}.$$

Proof. Since $\{X_t\}$ is a strong Markov process, see Davis (1984), we need just to prove that the assertion holds for the first jump. The jump measure from a point $x \in \partial^* E$ is verified easily. Thus we suppose that the jump considered below is from the inside of E. Fix an initial point x and denote the time of the first jump by T_1. Because the path until T_1 is deterministic and $\widetilde{\mathbb{P}}_x$ is absolutely continuous with respect to $\mathbb{P}_x$, it follows that $\{X_t\}$ follows the vector-field χ until T_1. Let t be such that $\phi(x, s), s \le t$ does not reach the boundary. Then for a Borel set B of $E \cup \{\Delta\}$

$$\begin{aligned}
&\widetilde{\mathbb{P}}_x[T_1 \le t, X_{T_1} \in B] \\
&\quad = \psi(x)^{-1}\int_0^t \lambda(\phi(x, s)) \exp\left\{-\int_0^s \lambda(\phi(x, v))\mathrm{d}v\right\} \\
&\qquad \times \int_B \psi(y)Q(\phi(x, s), \mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

Thus the distribution of the point after the jump depends on the point $\phi(x, T_1)$ prior to the jump only. This gives $\tilde{Q}$. We need to determine the jump intensity and to show that it is "independent" of the starting point x. The time of the first jump has density

$$\begin{aligned}
\tilde{f}_{T_1}(t) &= \psi(x)^{-1}\lambda(\phi(x, t)) \exp\left\{-\int_0^t \lambda(\phi(x, v))\mathrm{d}v\right\} \\
&\qquad \times \int_{E\cup\{\Delta\}} \psi(y)Q(\phi(x, t), \mathrm{d}y).
\end{aligned}$$