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Characterization of upper comonotonicity via tail convex order

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ABSTRACT

In this paper, we show a characterization of upper comonotonicity via tail convex order. For any given marginal distributions, a maximal random vector with respect to tail convex order is proved to be upper comonotonic under suitable conditions. As an application, we consider the computation of the Haezendonck risk measure of the sum of upper comonotonic random variables with exponential marginal distributions.

1. Introduction

In the study of the riskiness of an insurance portfolio, we are often interested in the total loss as a sum of risk variables. While traditional approaches have been based on the independence assumption among risks, recent trends allow dependent structures so that we can model the real situation more plausibly. Complications due to dependence among risks can be avoided by, or appropriately dealt with through, replacing the original sum by a less attractive one with a simpler dependence structure. Several concepts have been introduced to compare risks in the actuarial literature (see e.g. Denuit et al., 2005; Dhaene et al., 2006). One common way is to consider convex order which uses stop-loss premiums with the same mean.

It is well known that if a random vector with given marginal distributions is comonotonic, then it has the largest sum with respect to convex order (see e.g. Dhaene et al., 2002a). A proof based on geometric interpretation of the comonotonic support can be found in Kaas et al. (2002). Conversely, it is also true that if a sum is maximal in convex order in the set of all random vectors with predetermined marginal distributions, then the underlying random variables must be comonotonic. Cheung proved the claim for the case of continuous marginal distributions in Cheung (2008). Later it was generalized to the case of integrable distributions in Cheung (2010). In Mao and Hu (2011) the authors showed that the problem boils down to a bivariate case and the joint distribution is indeed the minimum of the two marginal distributions. The well-known fact that a random vector is comonotonic if and only if it is pairwise comonotonic is used. It is, however, no longer valid for an upper comonotonic random vector and a counter-example is given in the Appendix of this paper.

The concept of upper comonotonicity was introduced and investigated in Cheung (2009). A characterization using joint distributions and the additivity of value at risk, tail value at risk, and expected shortfall were given. In Dong et al. (2010), the authors proved the additivity of α-mixed inverse distribution functions and stop-loss premiums for the sum of upper comonotonic random variables. In Cheung (2010), it was shown that the additivity of value at risk for the confidence level being sufficiently large implies the upper comonotonicity.

On the other hand, as in the comonotonic case, we can consider an ordering between the sum S of the components of a random vector $\mathbf{X}$ and the corresponding sum S^{uc} of an upper comonotonic

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random vector $\mathbf{X}^{uc}$. Indeed, if $\mathbf{X}^{uc}$ coincides with $\mathbf{X}$ in the lower tail in distribution and they have the same marginal distributions, then the sum S precedes S^{uc} in convex order (for details, see Dong et al., 2010). If there is no coincidence in the lower tail, then the convex order is no longer valid in general and it should be relaxed to describe the upper tail only.

The concept of tail convex order was introduced by Cheung and Vanduffel (forthcoming), who proved that, upon suitable conditions, the sum of the components of an upper comonotonic random vector is the largest in tail convex order. This suggests that upper comonotonicity be related to tail convex order rather than convex order. Therefore, the following characterization question arises: If the sum of the components of a random vector is maximal in tail convex order, then is this random vector upper comonotonic? In the present paper, we shall focus on this question and pursue an answer.

The rest of this paper is organized as follows: Section 2 prepares some necessary definitions and states our main result, Section 3 formulates the proof of the result, Section 4 shows an application to the computation of Haezendonck risk measures, and, finally, Appendix contains some examples to further clarify some fallacies.

2. Main result

The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$, where real valued random variables or risks X_i and Y_i are defined for $i = 1, \dots, n$.

A subset $C \subseteq \mathbb{R}^n$ is said to be comonotonic if for any $\mathbf{x}$ and $\mathbf{y}$ in C, either $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{y} \leq \mathbf{x}$ holds. We call a random vector $\mathbf{X} = (X_1, \dots, X_n)$ to be comonotonic if it has a comonotonic support.

As a generalization of comonotonicity, we consider upper comonotonicity. For any given $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{R} \cup \{-\infty\})^n$, the upper quadrant $(d_1, \infty) \times \cdots \times (d_n, \infty)$ is denoted by $U(\mathbf{d})$, the lower quadrant $(-\infty, d_1] \times \cdots \times (-\infty, d_n]$ by $L(\mathbf{d})$, and the remaining complement $\mathbb{R}^n \setminus (U(\mathbf{d}) \cup L(\mathbf{d}))$ by $R(\mathbf{d})$.

Definition 2.1 (*Cheung, 2009*)**.** A random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be upper comonotonic if there exist some $\mathbf{d} \in (\mathbb{R} \cup \{-\infty\})^n$ and some null set N such that

(i) $\{\mathbf{X}(\omega) : \omega \in \Omega \setminus N\} \cap U(\mathbf{d})$ is comonotonic,
(ii) $P(\mathbf{X} \in U(\mathbf{d})) > 0$,
(iii) $\{\mathbf{X}(\omega) : \omega \in \Omega \setminus N\} \cap R(\mathbf{d})$ is empty.

The usual inverse distribution functions $F_X^{-1}(p)$, $F_X^{-1+}(p)$ of a random variable $X : \Omega \to \mathbb{R}$ are defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\},$$

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} : F_X(x) \leq p\},$$

respectively, with the convention $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. To pick up any point in the closed interval $[F_X^{-1}(p), F_X^{-1+}(p)]$, we introduce the so-called α-mixed inverse distribution functions of F_X as

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1 - \alpha) F_X^{-1+}(p), \quad p \in (0, 1).$$

See e.g. Dhaene et al. (2002a) for related discussions. From now on, the distribution function F_{X_i} or F_{Y_i} will be denoted by F_i as long as no confusion arises.

A random variable X is said to precede another random variable Y in stop-loss order (written as $X \leq_{sl} Y$) if X has less stop-loss premiums than Y, i.e. $\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$ for all $d \in \mathbb{R}$. It is well known that stop-loss order preserves the ordering of TVaR_p and vice versa, i.e. $X \leq_{sl} Y$ if and only if $\mathrm{TVaR}_p[X] \leq \mathrm{TVaR}_p[Y]$ for all $p \in (0, 1)$ (see Dhaene et al., 2006).

A random variable X is said to precede Y in convex order (written as $X \leq_{cx} Y$) if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $X \leq_{sl} Y$. For an overview of recent progresses on comonotonicity-based convex bounds, see Dhaene et al. (2002b), Deelstra et al. (2011), and references therein. In particular, by limiting the range of d in stop-loss order, we are led to the following definition.

Definition 2.2 (*Cheung and Vanduffel, forthcoming*)**.** A random variable X is said to precede another random variable Y in tail convex order (written as $X \leq_{tcx} Y$) if there exists some d^* such that

(i) $P(X \geq d^*) > 0$,
(ii) $\mathbb{E}[X] = \mathbb{E}[Y]$,
(iii) $\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$ for all $d \geq d^*$.

For any given $\mathbf{Y}$, the existence of an upper comonotonic random vector $\mathbf{Y}^{uc}$ satisfying $S \leq_{tcx} S^{uc}$ is shown in Section 4 of Dong et al. (2010), where (here and throughout) S and S^{uc} are the sums of the components of $\mathbf{Y}$ and $\mathbf{Y}^{uc}$, respectively.

For any given distribution functions $F_1, \dots, F_n$, the Fréchet space $\mathcal{R} = \mathcal{R}(F_1, \dots, F_n)$ is defined to be the set of all random vectors with marginal distributions $F_1, \dots, F_n$. Note that if we restrict ourselves on $\mathcal{R}$, then stop-loss order is exactly the same as convex order.

Now we are ready to state the main result. In what follows, for any given $A \subset \mathbb{R}^n$, its closure, interior, and boundary are denoted by $\overline{A}$, $\mathrm{int}(A)$, and $\mathrm{bd}(A)$, respectively.

Theorem 2.3. *For any given marginal distributions $F_1, \dots, F_n$, if there exists $\mathbf{Y} = (Y_1, \dots, Y_n) \in \mathcal{R} = \mathcal{R}(F_1, \dots, F_n)$ such that*

$$X_1 + \cdots + X_n \leq_{tcx} Y_1 + \cdots + Y_n, \quad \forall (X_1, \dots, X_n) \in \mathcal{R}, \tag{1}$$

then there exist some $\mathbf{d}^* \in \mathbb{R}^n$ and a null set $N \subset \Omega$ such that

(i) *$P(\mathbf{Y} \in \overline{U(\mathbf{d}^*)}) > 0$,*
(ii) *$\{\mathbf{Y} : \omega \in \Omega \setminus N\} \cap \overline{U(\mathbf{d}^*)}$ is comonotonic,*
(iii) *$\{\mathbf{Y} : \omega \in \Omega \setminus N\} \cap \mathrm{int}(R(\mathbf{d}^*)) = \emptyset$.*

Furthermore, if $P(\mathbf{Y} \in \mathrm{bd}(U(\mathbf{d}^*))) = 0$, then $(Y_1, \dots, Y_n)$ is upper comonotonic with a threshold in $L(\mathbf{d}^*)$.

Note that (i)–(iii) of the theorem are similar to, but not exactly the same as, the three conditions in Definition 2.1. Example A.2 below shows the necessity of the condition $P(\mathbf{Y} \in \mathrm{bd}(U(\mathbf{d}^*))) = 0$ for $\mathbf{Y}$ to be upper comonotonic.

3. Proof of the theorem

The proof of Theorem 2.3 will be given by observing relations between $\mathbf{Y}$ and its comonotonic counterpart $\mathbf{Y}^c$ in terms of stop-loss premiums.

3.1. Lemmas

To begin with, we recall the following lemma:

Lemma 3.1 (*Dhaene et al., 2002a*)**.** *The stop-loss premiums of the sum S^c of the components of a comonotonic random vector $\mathbf{Y}^c = (Y_1^c, \dots, Y_n^c)$ are given by*

$$\mathbb{E}[(S^c - d)_+] = \sum_{i=1}^{n} \mathbb{E}[(Y_i^c - d_i)_+] \quad \textit{for } F_{S^c}^{-1+}(0) < d < F_{S^c}^{-1}(1),$$

where $d_i = F_i^{-1(\alpha)}(F_{S^c}(d))$ and α solves the equation $F_{S^c}^{-1(\alpha)}(F_{S^c}(d)) = d$.

Note that a similar result also holds true for an upper comonotonic random vector (see e.g. Dong et al., 2010). In addition, we can see that $d_i = F_i^{-1(\alpha)}(q)$ is defined only for values of the form $q = F_{S^c}(d)$. Nevertheless, the applicability of Lemma 3.1 during the proof of Theorem 2.3 is guaranteed by the following lemma, which shows that the image of each F_i is a subset of the image of F_{S^c}, namely, $\mathrm{Im}(F_i) \subseteq \mathrm{Im}(F_{S^c})$.

Lemma 3.2. *For any random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$ and its comonotonic counterpart $\mathbf{Y}^c$, we have*

$$\mathrm{Im}(F_i) \subseteq \mathrm{Im}(F_{S^c}), \quad i = 1, \dots, n.$$