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# Mathematical investigation of the Gerber–Shiu function in the case of dependent inter-claim time and claim size

Éva Orbán Mihálykó *, Csaba Mihálykó

*University of Pannonia, Department of Mathematics, H-8200 Veszprém, Egyetem u. 10, Hungary*

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ABSTRACT

In this paper we investigate the well-known Gerber–Shiu expected discounted penalty function in the case of dependence between the inter-claim times and the claim amounts. We set up an integral equation for it and we prove the existence and uniqueness of its solution in the set of bounded functions. We show that if $\delta > 0$, the limit property of the solution is not a regularity condition, but the characteristic of the solution even in the case when the net profit condition is not fulfilled. It is the consequence of the choice of the penalty function for a given density function. We present an example when the Gerber–Shiu function is not bounded, consequently, it does not tend to zero. Using an operator technique we also prove exponential boundedness.



## 1. Introduction

In risk theory the surplus process is given by

$$U(t) = x + ct - \sum_{k=1}^{N(t)} Y_k,$$

where $x \geq 0$ is the initial surplus, $c$ is the insurer's rate of the premium income per unit time, $N(t)$ is the number of claims up to time $t$, and $Y_k$ $k = 1, 2, 3, \ldots$ is the size of the $k$th claim size.

The main topics of interest are the probability of ruin, the expectation of the time of ruin, the distribution of the surplus before ruin and the deficit at ruin. To answer these questions and to investigate the characteristics of the surplus process, the Gerber–Shiu expected discounted penalty function was introduced in Gerber and Shiu (1997, 1998). Many papers deal with the determination of this function (for example Gerber and Shiu (2005) and Albrecher et al. (2010)). Integral and integro-differential equations have been set up and solved in special cases and the properties of this function are also of interest.

Risk theory describing the surplus process usually relies on the assumption of independence between the inter-claim times and claim sizes, although it is too restrictive for the real world. Recently some results have been published for models that allow for specific dependence between claim size and claim-time (Albrecher and Teugels, 2006; Boudreault et al., 2006; Ambagaspitiya, 2009).

In this paper we deal with the well-known Gerber–Shiu expected discounted penalty function and prove that some properties of the solution (namely its limit and exponential convergence) are the consequence of the equation under certain condition. In Albrecher et al. (2010) the authors mention that due to the net profit condition the expected discounted penalty function tends to zero if its first argument tends to infinity. They considered this limit property as a regularity condition of the equation. We show that the value of the limit is the consequence of the choice of the penalty function $w$ and density function $h$ in the case $\delta > 0$. We present an example when $\lim_{x\to\infty} m_\delta(x) \neq 0$, moreover another one, when the function $m_\delta(x)$ is not bounded. We give a necessary and sufficient condition for functions $w$ and $h$ which assures that the limit of the bounded Gerber–Shiu discounted penalty function is zero if $x \to \infty$.

Let us consider the risk model with the following notations: $t_0 = 0$, $t_k$, $k = 1, 2, 3, \ldots$ are the inter-arrival times, $Y_k$ is the $k$th claim size, $(t_k, Y_k)$ has common distribution function $H(t, y)$ and density function $h(t, y)$ for any value of $k = 1, 2, 3, \ldots$.

* Corresponding author. Tel.: +36 88 624122.
*E-mail address:* orbane@almos.uni-pannon.hu (É. Orbán Mihálykó).

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We suppose that $(t_k, Y_k)$ are independent for $k = 1, 2, 3, \ldots$ but the random variables $t_k$ and $Y_k$ may be dependent for fixed values of $k$. This model is a generalization of the well-known Sparre Andersen risk process. The marginal distribution function of $t_k$ is denoted by $F(t)$, its density function by $f(t)$, where $t_k$ is nonnegative with finite expectation $\mu_f$. The marginal distribution function of $Y_k$ is denoted by $G(y)$, its density function by $g(y)$, where $Y_k$ is nonnegative with finite expectation $\mu_g$. The number of claims up to time $t$ is given by

$$N(t) = \begin{cases} 0 & \text{if } t_1 < t, \\ n & \text{if } \sum_{k=1}^{n} t_k < t \text{ and } \sum_{k=1}^{n+1} t_k \geq t. \end{cases}$$

The ultimate probability of ruin is

$$\psi(x) = P\left(x + ct - \sum_{k=1}^{N(t)} Y_k < 0 \quad \text{for some } t \geq 0\right).$$

The time of ruin is

$$T = \begin{cases} \inf\{t : U(t) < 0\} & \text{if there exists } t \geq 0 \text{ such that } U(t) < 0, \\ \infty & \text{if } U(t) \geq 0 \text{ for all } t \geq 0. \end{cases}$$

Let $w : R_0^+ x R_0^+ \to R_0^+$ be the penalty function.

The Gerber–Shiu expected discounted penalty function is defined as

$$m_\delta(x) = E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty} \mid U(0) = x), \tag{1}$$

if the expectation is finite. Here $x \geq 0$, $\delta \geq 0$ and $U(T^-)$ is the surplus immediately before ruin, $|U(T)|$ is the deficit at ruin. Note that $0 \leq m_\delta(x)$, and it is monotone decreasing in $\delta$. For the sake of simplicity we suppose that $h$ and $w$ are continuous functions on $[0, \infty)x[0, \infty)$.

In this paper we analyze some properties of the function $m_\delta(x)$. We emphasize that these properties are also valid without the usual net profit condition $(0 < c\mu_f - \mu_g)$.

The paper is organized as follows: In Section 2 we present the integral equation satisfied by $m_\delta(x)$, and we prove the existence and uniqueness of the solution in the set of bounded functions. In Section 3 we deal with the limit of the bounded solution $m_\delta(x)$ and in Section 4 we give examples for bounded and unbounded Gerber–Shiu functions $m_\delta(x)$. In Section 5 we investigate the exponential bounds and the role of the net profit condition. Finally, we summarize our results in Section 6.

## 2. The equation satisfied by the Gerber–Shiu expected discounted penalty function

Using the well-known renewal technique we prove

**Theorem 1.** *Suppose* $S(x) = \int_0^\infty \int_{x+ct}^\infty \mathrm{e}^{-\delta t} w(x + ct, y - ct - x) h(t, y)\mathrm{d}y\mathrm{d}t < \infty$ *for all* $x \geq 0$, *then the function* $m_\delta(x)$ *satisfies the following integral equation:*

$$m_\delta(x) = \int_0^\infty \int_0^{x+ct} \mathrm{e}^{-\delta t} m_\delta(x + ct - y) h(t, y)\mathrm{d}y\mathrm{d}t + \int_0^\infty \int_{x+ct}^\infty \mathrm{e}^{-\delta t} w(x + ct, y - ct - x) h(t, y)\mathrm{d}y\mathrm{d}t. \tag{2}$$

**Proof.** For the sake of clarity we omit the indication of the condition $U(0) = x$ in the proof.

Take the conditional expectation on the first claim time and its size, given $t_1 = t$ and $Y_1 = y$.

$$\begin{aligned} m_\delta(x) &= E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty}) \\ &= E(E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty} | t_1, Y_1)) \\ &= \int_0^\infty \int_0^\infty E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty} | t_1 = t, Y_1 = y) \\ &\quad \cdot h(t, y)\mathrm{d}y\mathrm{d}t. \end{aligned}$$

To determine $E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty} | t_1 = t, Y_1 = y)$, two different cases are distinguished:
a. If $y \leq x + ct$, the process is renewed at the first claim. The initial surplus of the renewed process is $x + ct - y$. The expectation of the time of ruin of the renewed process is less then the expectation of the time to ruin of the original surplus process, their difference is $t$. Consequently, if $y \leq x + ct$, then

$$\begin{aligned} &E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty} | t_1 = t, Y_1 = y) \\ &\quad = \mathrm{e}^{-\delta t} m_\delta(x + ct - y). \end{aligned}$$

b. If $y > x + ct$, ruin occurs at the first claim, consequently $T = t$. The surplus before ruin equals $x + ct$, the deficit at ruin is $y - ct - x$. Consequently, if $y > x + ct$, then

$$\begin{aligned} &E(\mathrm{e}^{-\delta T} w(U(T^-), |U(T)|) \cdot 1_{T<\infty} | t_1 = t, Y_1 = y) \\ &\quad = \mathrm{e}^{-\delta t} w(x + ct, y - ct - x). \end{aligned}$$

Applying these equalities we get Eq. (2). □

It is easy to see that Eq. (2) is a generalized form of (1) in Albrecher et al. (2010).

We will investigate Eq. (2) from the mathematical point of view. We distinguish the solution of the equation and the function defined by (1), hence the solutions of the equation will be denoted by $\phi_\delta(x)$, and the equation has the form

$$\phi_\delta(x) = \int_0^\infty \int_0^{x+ct} \mathrm{e}^{-\delta t} \phi_\delta(x + ct - y) h(t, y)\mathrm{d}y\mathrm{d}t + \int_0^\infty \int_{x+ct}^\infty \mathrm{e}^{-\delta t} w(x + ct, y - ct - x) h(t, y)\mathrm{d}y\mathrm{d}t. \tag{3}$$

The distinction is interesting in the case when the solution of Eq. (3) is not unique. By Eq. (2), we know that $m_\delta(x)$ is a solution of Eq. (3), but Eq. (3) may have other solutions as well.

In Albrecher et al. (2010), the authors seek the solution of the equation concerning $m_\delta(x)$ in the set of continuous functions vanishing at infinity assuming independence. This set is a subset of the bounded functions. First we investigate the existence and uniqueness of the solution of Eq. (3) in the set of bounded functions.

**Theorem 2.** *Let* $\delta > 0$ *be fixed. Suppose that*

$$\begin{aligned} \kappa &= \sup_{x \geq 0} \int_0^\infty \int_{x+ct}^\infty \mathrm{e}^{-\delta t} w(x + ct, y - ct - x) h(t, y)\mathrm{d}y\mathrm{d}t \\ &= \sup_{x \geq 0} S(x) < \infty. \end{aligned} \tag{A}$$

*Then Eq. (3) has a unique solution in the set of bounded functions on* $[0, \infty)$.

**Proof.** Consider the set of bounded and continuous functions on $[0, \infty)$ with supremum norm $\|.\|$. Let us define the operator$K_\delta$ as follows: $K_\delta$ maps from the set of bounded and continuous functions of one variable to the set of bounded and continuous functions of one variable applying the following definition

$$K_\delta(\phi)(x) = \int_0^\infty \int_0^{x+ct} \mathrm{e}^{-\delta t} \phi(x + ct - y) h(t, y)\mathrm{d}y\mathrm{d}t + \int_0^\infty \int_{x+ct}^\infty \mathrm{e}^{-\delta t} w(x + ct, y - ct - x) h(t, y)\mathrm{d}y\mathrm{d}t,$$

where $\phi$ is a bounded and continuous function of one variable.