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Optimal consumption and portfolio policies with the consumption habit constraints and the terminal wealth downside constraints☆

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ABSTRACT

In this paper, we consider the optimal consumption and portfolio policies with the consumption habit constraints and the terminal wealth downside constraints, that is, here the consumption rate is greater than or equal to some nonnegative process, and the terminal wealth is no less than some positive constant. Using the martingale approach, we get the optimal consumption and portfolio policies.

1. Introduction

Optimal intertemporal consumption and portfolio policies in continuous time under uncertainty have traditionally been interesting. Merton (1971) introduced the dynamic programming method in order to study the optimal consumption and portfolio selection problem in continuous time. Cox and Huang (1989); Karatzas and Shreve (1998) and Karatzas and Shreve (1991) introduced the martingale method independently.

There is no restriction on computation in the above two papers, except the nonnegative consumption and nonnegative terminal wealth constraints. Recently, portfolio problems with these kinds of constraints have been remarkably studied. Usually, these constraints include consumption habit constraints (or consumption constraints) and downside terminal wealth constraints.

For consumption constraints which many investors will agree with, one simple example is the endowment funds; see Thaler and Williamson (1994) and Cheng and Wei (2005). Meanwhile, from the view point of the real life, it is also generally accepted that people have the nature of keeping and improving their living standards. They will be unhappy and try to overcome the relatively hard time if the consumptions are below their own habit levels. A reasonable assumption, in our opinion, is to consider the consumption habit as a consumption constraint, $c(t) \geq h(t)$, where $c(t)$ is the consumption level at time t, and $h(t)$ is the consumption habit at time t. Cheng and Wei (2005) considered the portfolio and consumption decisions with the consumption habit constraints, and obtained the optimal consumption behavior for the consumption habit using the martingale approach. For more references, see Abel (1990); Campbell and Cochrane (2000); Detemple and Karatzas (2003). If $h(t) \equiv R$, R is some positive constant, i.e. the consumption rate process is subjected to downside constraint, Shin et al. (2007) studied a general consumption and portfolio selection problem and obtained the general optimal policies in an explicit form. In Section 3 of Lakner and Nygren (2006), they discussed the portfolio optimization problem for an investor whose consumption rate process was also subjected to downside constraints.

In real life, portfolio problems often include a downside terminal wealth constraint. This could be a liquidity constraint or can be caused by a guaranteed lower bound on the capital. One simple example is the optimal investment in insurance mathematics where the return contains a guaranteed benefit plus a bonus, see Korn (2005). So, a reasonable assumption could be that $W \geq K$, where W is the final wealth, and K is a positive constant. This constraint is the so-called downside constraint (also called an insurance constraint). In Section 4 of Lakner and Nygren (2006), they discussed maximizing the expected utility from terminal wealth under the downside constraints.

In Section 5 of Lakner and Nygren (2006), they discussed the portfolio optimization problem for an investor whose consumption rate process and terminal wealth are subjected to downside constraints, i.e. $c(t) \geq R$, $W \geq K$. In their paper, consumption

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habit constraint does not depend on time t. It should be interesting to consider the case where the consumption habit constraint depends on time t, which means that $c(t) \geq h(t)$.

In this paper, we study the optimal consumption and portfolio policy with both consumption habit constraints and terminal wealth downside constraints. It should be mentioned that Lakner and Nygren (2006) considered the consumption rate is bounded below by a constant, and the price of risk securities follows geometric Brownian motion. Whereas, the model we are interested in here is that the consumption rate is bounded below by some function depending on time, and the price process for the risk securities follows an Itô process. Using the so-called martingale approach, we obtain the optimal consumption and portfolio policy.

2. The model

Fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a time span $[0, \mathbb{T}]$, where $\mathbb{T}$ is a strictly positive real number. Let $\omega = \{\omega_n(t) : t \in [0, \mathbb{T}],\ n = 1, 2, \dots N\}$ be an N dimensional standard Brownian motion defined on the probability space and $\{\mathbb{F} = \mathcal{F}_t, t \in [0, \mathbb{T}]\}$ be the filtration generated by ω.

We assume that $\mathbb{F}$ is complete in the sense that $\mathcal{F}_0$ contains all the P null sets and that $\mathcal{F}_{\mathbb{T}} = \mathcal{F}$. By the definition of N dimensional standard Brownian motion, $\omega(0) = 0$, a.s., so $\mathcal{F}_0$ is almost trivial.

We consider a security market which consists of $N+1$ securities traded, indexed by $n = 0, 1, 2, \dots, N$. Security $n \neq 0$ is risky and pays dividends at rate $l_n(t)$ and sells for $S_n(t)$ at time t. We will use $S(t)$ to denote $(S_1(t), \dots, S_N(t))^{\mathrm{T}}$, where T means that the transpose of a vector. Assume that $l_n(t)$ can be written as $l_n(S(t), t)$ with $l_n(y, t) : \mathbb{R}^{\mathbb{N}} \times [0, \mathbb{T}] \longrightarrow \mathbb{R}$ being Borel measurable.

Security $n = 0$ is (locally) riskless, pays no dividends, and sells for

$$B(t) = \exp\left\{\int_0^t r(s)\mathrm{d}s\right\}. \tag{2.1}$$

Assume further that $r(t) \triangleq r(S(t), t) : \mathbb{R}^{\mathbb{N}} \times [0, \mathbb{T}] \longrightarrow \mathbb{R}$ is continuous.

The price process for risk securities $S(t)$ follows an Itô process satisfying

$$S(t) + \int_0^t l(s)\mathrm{d}s = S(0) + \int_0^t \mu(S(s), s)\mathrm{d}s + \int_0^t \sigma(S(s), s)\mathrm{d}\omega(s), \quad \forall t \in [0, \mathbb{T}],\ \text{a.s.} \tag{2.2}$$

where l is a N vector of l_n's, $\mu(y, t) : \mathbb{R}^{\mathbb{N}} \times [0, \mathbb{T}] \longrightarrow \mathbb{R}^{\mathbb{N}}$ and $\sigma(y, t) : \mathbb{R}^{\mathbb{N}} \times [0, \mathbb{T}] \longrightarrow \mathbb{R}^{\mathbb{N}}$ are continuous in y and t.

Assumption 1. The diffusion matrix $\sigma(S(t), t)$ satisfies the non-degeneracy condition $y^{\mathrm{T}}\sigma(S(t), t)\sigma(S(t), t)^{\mathrm{T}} y \geq \varepsilon|y|^2$, almost surely for all $(y, t) \in \mathbb{R}^{\mathbb{N}} \times [0, \mathbb{T}]$ and some $\varepsilon > 0$.

This condition implies in particular that $\sigma(S(t), t)$ has full rank almost surely for all $t \in [0, \mathbb{T}]$.

Defining $\widetilde{S}(t) \triangleq \frac{S(t)}{B(t)}$, $\widetilde{l}(t) \triangleq \frac{l(t)}{B(t)}$, Itô's formula implies that

$$\begin{aligned}\widetilde{S}(t) + \int_0^t \widetilde{l}(s)\mathrm{d}s &= \widetilde{S}(0) + \int_0^t \frac{\mu(S(s), s) - r(S(s), s)S(s)}{B(s)}\mathrm{d}s + \int_0^t \frac{\sigma(S(s), s)}{B(s)}\mathrm{d}\omega(s)\\ &= \widetilde{S}(0) + \int_0^t \frac{\sigma(S(s), s)[\sigma(S(s), s)^{-1}(\mu(S(s), s) - r(S(s), s)S(s))\mathrm{d}s + \mathrm{d}\omega(s)]}{B(s)}.\end{aligned} \tag{2.3}$$

Define

$$G(t) \triangleq S(t) + \int_0^t l(s)\mathrm{d}s,$$

and

$$\widetilde{G}(t) \triangleq \widetilde{S}(t) + \int_0^t \widetilde{l}(s)\mathrm{d}s,$$

where the former is the N vector of gains processes, and the latter is the N vector of gain processes in units of the 0th security.

We shall use the following notation: If A is a matrix, then $|A|^2$ denotes $tr(AA^{\mathrm{T}})$ where tr means trace.

Define

$$\kappa(S(t), t) \triangleq -\sigma(S(t), t)^{-1}(\mu(S(t), t) - r(S(t), t)S(t)), \tag{2.4}$$

and

$$\eta(t) \triangleq \exp\left\{\int_0^{\mathbb{T}} \kappa^{\mathrm{T}}\mathrm{d}\omega(s) - \frac{1}{2}\int_0^{\mathbb{T}} |\kappa|^2\mathrm{d}s\right\}. \tag{2.5}$$

For a given $\mathbb{T} > 0$, we define the equivalent martingale measure $\widetilde{P}$ as follows

$$\widetilde{P}(A) \triangleq \mathbb{E}[\eta(\mathbb{T})I_A], \quad \forall A \in \mathcal{F}_{\mathbb{T}}, \tag{2.6}$$

then

$$\eta(t) = \mathbb{E}\left[\frac{\mathrm{d}\widetilde{P}}{\mathrm{d}P}|\mathcal{F}_t\right] \quad \forall t \in [0, \mathbb{T}],\ \text{a.s.} \tag{2.7}$$

We will use the $\widetilde{\mathbb{E}}$ to denote the expectation under $\widetilde{P}$.

By Girsanov's theorem, we know that the process

$$\widetilde{\omega}(t) = \omega(t) - \int_0^t \kappa(S(s), s)\mathrm{d}s \tag{2.8}$$

is a standard Brownian motion under $\widetilde{P}$, and so

$$\widetilde{G}(t) = \widetilde{S}(0) + \int_0^t \frac{\sigma(S(s), s)}{B(s)}\mathrm{d}\widetilde{\omega}(s) \tag{2.9}$$

is a local martingale under $\widetilde{P}$.

A trading strategy is an $N+1$ dimensional process (α, θ), where $\alpha(t)$ and $\theta_n(t)$ represents the number of units of the 0th security and the nth security, respectively, held at time t, and $\theta(t)$ is measurable, adapted and satisfies $\widetilde{\mathbb{E}}[\int_0^{\mathbb{T}} \theta(t)^{\mathrm{T}}\sigma(S(s), s)\sigma(S(s), s)^{\mathrm{T}} \theta(t)\mathrm{d}t] < \infty$.

If there exists a consumption plan (c, W) such that for every $t \in [0, \mathbb{T}]$

$$\alpha(t)B(t) + \theta(t)^{\mathrm{T}}S(t) + \int_0^t c(s)\mathrm{d}s = \alpha(0)B(0) + \theta(0)^{\mathrm{T}}S(0) + \int_0^t \alpha(s)\mathrm{d}B(s) + \int_0^t \theta(s)^{\mathrm{T}}\mathrm{d}G(s),\ \text{a.s.} \tag{2.10}$$

and

$$W = \alpha(\mathbb{T})B(\mathbb{T}) + \theta(\mathbb{T})^{\mathrm{T}}S(\mathbb{T}),\ \text{a.s.} \tag{2.11}$$

where $\{c(t), 0 \leq t \leq \mathbb{T}\}$ is a consumption rate process which is nonnegative, progressive, measurable, and satisfies $\int_0^{\mathbb{T}} c(t)\mathrm{d}t < \infty$, a.s.. We call (α, θ, c, W) is a self-financing strategy. Let H denote the space of all self-financing strategies. It is very easy to verify that H is a linear space.

Lemma 2.1. *Let $(\alpha, \theta, c, W) \in H$, define $M(t) \triangleq \alpha(t) + \theta(t)^{\mathrm{T}}\widetilde{S}(t) + \int_0^t \frac{c(s)}{B(s)}\mathrm{d}s$, $t \in [0, \mathbb{T}]$. Then $\{M(t), t \in [0, \mathbb{T}]\}$ is a local martingale under $\widetilde{P}$. Furthermore, it is a supmartingale.*