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On "optimal pension management in a stochastic framework" with exponential utility

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ABSTRACT

This paper reconsiders the optimal asset allocation problem in a stochastic framework for defined-contribution pension plans with exponential utility, which has been investigated by Battocchio and Menoncin [Battocchio, P., Menoncin, F., 2004. Optimal pension management in a stochastic framework. Insurance: Math. Econ. 34, 79–95]. When there are three types of asset, cash, bond and stock, and a non-hedgeable wage risk, the optimal pension portfolio composition is horizon dependent for pension plan members whose terminal utility is an exponential function of real wealth (nominal wealth-to-price index ratio). With market parameters usually assumed, wealth invested in bond and stock increases as retirement approaches, and wealth invested in cash asset decreases. The present study also shows that there are errors in the formulation of the wealth process and control variables in solving the optimization problem in the study of Battocchio and Menoncin, which render their solution erroneous and lead to wrong results in their numerical simulation.

1. Introduction

Defined-contribution (DC) pension plans have achieved growing popularity among corporate sponsors. Studies on the optimal asset allocation strategy for DC pension plans generally assume that pension plan members have power utility (Boulier et al., 2001; Cairns et al., 2006; Deelstra et al., 2003) or exponential utility (Battocchio and Menoncin, 2004; Henderson, 2005) over terminal wealth (or some variables derived from terminal wealth). With power utility, analytical solution usually does not exist when there is non-hedgeable wage risk. With exponential utility, analytical solutions can be derived for scenarios where non-hedgeable wage risk exists. Although power utility is generally assumed to be more consistent with empirical data, in the opinion of Henderson (2005), Bliss and Panigirtzoglou (2004) have found evidence from option prices that exponential utility provides better representation of preferences than power utility.

Battocchio and Menoncin (2004) have derived a solution for the optimal asset allocation problem for DC pension plans with salary risk and inflation risk. In their model, the objective of the pension plan is to maximize the expected utility which is a negative exponential function of the real terminal wealth, and there are three asset categories; a riskless asset, bonds and stocks. However, their solution appears to have some problems in terms of model formulation and control variables. Their formulation implicitly assumes that allocation to all the three asset types can be determined independently, but in fact only two assets can be determined independently (see Appendix A for detailed comments). Once investments in two assets have been set, the third one is automatically set. That implicit assumption leads to an incorrect solution, which is illustrated by their numerical simulation results (Battocchio and Menoncin 2004).

The aim of the present paper is to address the same optimal asset allocation issue in a stochastic framework with an improved formulation of wealth process and control variables. The rest of the paper is organized as follows. Section 2 will present a formulation of real wealth process where only two asset proportions can be independently determined. Section 3 will solve analytically the

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optimal asset allocation problem with the new formulation of wealth process, and provide a numerical simulation of pension wealth growth using the optimal allocation strategy. Section 4 will conclude.

2. The real wealth process and optimal asset allocation problem

The definitions and parameters are the same as those in the study by Battocchio and Menoncin (2004). Briefly, there are three types of asset in the financial market: cash, bonds and equities. The randomness in the financial market is described by two standard and independent Brownian motions $W_r(t)$ and $W_S(t)$ with $t \in [0, T]$. The instantaneous interest rate follows an Ornstein–Uhlenbeck process (Vasicek, 1977),

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW_r(t),$$
$$r(0) = r_0, \tag{1}$$

where α, β, and σ are strictly positive constants; σ is interest rate volatility. The changes in interest rate are mean-reverting where β is the "mean" level and α measures the strength of discrepancy driving the current rate back to that level.

The price process of riskless asset is given by

$$dS^0(t) = S^0(t)r(t)dt,$$
$$S^0(0) = S^0_0. \tag{2}$$

The return of a zero coupon bond with maturity $\tau \in [0, T]$ is given by

$$\frac{dB(t,\tau,r)}{B(t,\tau,r)} = [(r(t) + a(t,\tau)\sigma\lambda)dt - a(t,\tau)\sigma dW_r(t)], \tag{3}$$

where $a(t,\tau) = \frac{1-e^{-\alpha(\tau-t)}}{\alpha}$, which is an interest rate volatility scale factor.

A bond rolling over zero coupon bonds with constant time to maturity (τ_K) has a price process

$$\frac{dB_K(t,r)}{B_K(t,r)} = [(r(t) + a_K\sigma\lambda)dt - a_K\sigma dW_r(t)], \tag{4}$$

where $a_K = \frac{1-e^{-\alpha\tau_K}}{\alpha}$, which is the interest rate volatility scale factor for the rolling bond with constant time to maturity τ_K.

The process of total return on the risky asset stock is given by

$$dS(t,r) = S(t,r)[(r(t) + m_S)dt + \upsilon\sigma dW_r(t) + \sigma_S dW_S(t)],$$
$$S(0) = S_0. \tag{5}$$

The parameter m_S is interpreted as a risk premium; υ is volatility scale factor measuring how the volatility of interest rate affects the stock price; and σ_S is stock specific volatility.

The plan member's wage, $L(t)$, evolves according to the stochastic differential equation (SDE)

$$dL(t) = L(t)[(m_L + r(t))dt + k_r\sigma dW_r(t) + k_S\sigma_S dW_S(t) + \sigma_L dW_\pi(t)],$$
$$L(0) = L_0. \tag{6}$$

W_π is also an independent Brownian motion; σ_L is a non-hedgeable wage volatility; k_r and k_S are two volatility scale factors measuring how the risk sources of interest rate and stock affect the wages; and m_L can be considered as a wage premium. Each employee puts a constant proportion γ of her salary into the personal pension fund, $C(t) = \gamma L(t)$.

Battocchio and Menoncin (2004) assume that the terminal utility is a function of real wealth (wealth-to-price index ratio), and the price index process is governed by the SDE

$$dp(t) = p(t)[(m_\pi + r(t))dt + \rho_r\sigma dW_r(t) + \rho_S\sigma_S dW_S(t) + \sigma_\pi dW_\pi(t)],$$
$$p(0) = 1. \tag{7}$$

The parameters ρ_r and ρ_S are two scale factors measuring how the volatility of interest rate and stock affect the price index and σ_π is the inflation own volatility.

Let ϕ_0, ϕ_S and ϕ_B be the proportions of pension wealth invested in the riskless asset, bonds and stocks respectively, $\phi_0 + \phi_S + \phi_B = 1$. Since only two proportions can be determined separately and the effect of wage contribution is $\gamma L(t)dt$, the SDE for the nominal wealth process can be written as

$$dF_N = F_N\left[(1 - \phi_B - \phi_S)\frac{dS^0}{S^0} + \phi_S\frac{dS}{S} + \phi_B\frac{dB}{B}\right] + \gamma L(t)dt. \tag{8}$$

After substituting the values of the differentials, dF_N can be written as

$$dF_N = [F_N(r + \phi_S m_S + \phi_B a_K\sigma\lambda) + \gamma L]dt + F_N(\phi_S\upsilon - \phi_B a_K)\sigma dW_r + F_N\phi_S\sigma_S dW_S. \tag{9}$$

Let real wealth $F = \frac{F_N}{p}$, by applying the Ito's lemma we have $dF(t) = \frac{1}{p}dF_N - \frac{F_N}{p^2}dp + \frac{F_N}{p^3}(dp)^2 - \frac{1}{p^2}(dF_N dp)$. When the differentials are substituted, the SDE governing the real wealth process becomes

$$dF = (\phi' MF + u_1 F + u_2)dt + (\phi'\Gamma' + \Lambda')FdW, \tag{10}$$

where,

$$\phi \equiv \begin{bmatrix}\phi_S & \phi_B\end{bmatrix}',$$
$$M \equiv \begin{bmatrix} m_S - \rho_r^2\upsilon\sigma^2 - \rho_S\sigma_S^2 \\ a_K\sigma\lambda + a_K\rho_r\sigma^2 \end{bmatrix},$$
$$u_1 = \rho_r^2\sigma^2 + \rho_S^2\sigma_S^2 + \sigma_\pi^2 - m_\pi,$$
$$u_2 \equiv \frac{\gamma}{p}L,$$
$$\Gamma' \equiv \begin{bmatrix} \upsilon\sigma & \sigma_S & 0 \\ -a_K\sigma & 0 & 0 \end{bmatrix},$$
$$\Lambda \equiv \begin{bmatrix} -\rho_r\sigma & -\rho_S\sigma_S & -\sigma_\pi \end{bmatrix}'$$
$$W \equiv \begin{bmatrix} W_r & W_S & W_\pi \end{bmatrix}'. \tag{11}$$

The optimal asset allocation problem is the same as that in Battocchio and Menoncin (2004),

$$\max_{\phi} E_0[U(F(T), T)],$$

subject to

$$d\begin{bmatrix} z \\ F \end{bmatrix} = \begin{bmatrix} \mu_z \\ \phi' MF + u_1 F + u_2 \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ \phi'\Gamma' F + \Lambda' F \end{bmatrix} dW,$$
$$z(0) = z_0, \quad F(0) = F_0, \quad \forall\, 0 \le t \le T, \tag{12}$$

where,

$$\underset{3\times 1}{z} \equiv \begin{bmatrix} r & L & p \end{bmatrix}',$$
$$\underset{3\times 1}{\mu_z} \equiv \begin{bmatrix} \alpha(\beta - r) & L\mu_L & p\mu_\pi \end{bmatrix}',$$
$$\underset{3\times 3}{\Omega'} \equiv \begin{bmatrix} \sigma & 0 & 0 \\ Lk_r\sigma & Lk_S\sigma_S & L\sigma_L \\ p\rho_r\sigma & p\rho_S\sigma_S & p\sigma_\pi \end{bmatrix}.$$

The Hamiltonian corresponding to the optimization problem (12) is

$$H(J) = J_t + \mu' z\frac{\partial J}{\partial z} + (\phi' MF + u_1 F + u_2)\frac{\partial J}{\partial F} + \frac{1}{2}tr\left(\Omega'\Omega\frac{\partial^2 J}{\partial z^2}\right) + (\phi'\Gamma' + \Lambda')\Omega F\frac{\partial^2 J}{\partial z\partial F} + \frac{1}{2}(\phi'\Gamma'\Gamma\phi + 2\phi'\Gamma'\Lambda + \Lambda'\Lambda)F^2\frac{\partial^2 J}{\partial F^2}. \tag{13}$$