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# Portfolio selection problem with multiple risky assets under the constant elasticity of variance model

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ABSTRACT

This paper focuses on the constant elasticity of variance (CEV) model for studying the utility maximization portfolio selection problem with multiple risky assets and a risk-free asset. The Hamilton–Jacobi–Bellman (HJB) equation associated with the portfolio optimization problem is established. By applying a power transform and a variable change technique, we derive the explicit solution for the constant absolute risk aversion (CARA) utility function when the elasticity coefficient is −1 or 0. In order to obtain a general optimal strategy for all values of the elasticity coefficient, we propose a model with two risky assets and one risk-free asset and solve it under a given assumption. Furthermore, we analyze the properties of the optimal strategies and discuss the effects of market parameters on the optimal strategies. Finally, a numerical simulation is presented to illustrate the similarities and differences between the results of the two models proposed in this paper.



## 1. Introduction

The portfolio selection problem of utility maximization is a fundamental problem in mathematical finance and has inspired literally hundreds of researches. Merton (1969, 1971) proposed the stochastic control approach to study this investment problem for the first time. Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989) adapted the martingale approach to problems of utility maximization and much of this development appeared in Karatzas (1989) and Karatzas and Shreve (1991). Karatzas et al. (1991) investigated the utility maximization problem in an incomplete market and a same problem was considered in Zhang (2007). Recently, there has been much attention to an insurer's utility maximization problem which is usually studied via stochastic control theory; see Browne (1995) and Yang and Zhang (2005) for example. In Wang et al. (2007), closed-form strategies were obtained for different utilities maximization of an insurer through the martingale approach.

The above mentioned researches applying stochastic control theory generally assumed that the risky assets' prices are driven by geometric Brownian motions (GBMs). Although the studies using the martingale method provided results for risky assets with general price processes, they most find specific solutions for GBM model or similar ones merely. However, numerous studies (see e.g., Hobson and Rogers (1998) and the references therein) have shown that empirical evidences do not support the assumptions of GBM model in which the volatilities of risky assets' prices are deterministic. It is clear that a model with stochastic volatility will be more practical.

The constant elasticity of variance (CEV) model with stochastic volatility is a natural extension of the GBM model. This model received attention because it has the ability of capturing the implied volatility skew and can explain the volatility smile. The CEV model allows the volatility to change with the underlying price and was proposed by Cox and Ross (1976) for European option pricing. It was usually applied to pricing options, analyzing the sensitivities and implied volatilities of options; see e.g. Cox (1996), Lo et al. (2000), Davydov and Linetsky (2001), Detemple and Tian (2002), Jones (2003), Widdicks et al. (2005) and Hsu et al. (2008). Recently, Xiao et al. (2007) and Gao (2009a,b) have begun to apply the CEV model to the optimal investment research and investigated the utility maximization problem for a participant in the defined-contribution pension plan. Gu et al. (2010) used the CEV model for studying the optimal investment and reinsurance problems.

However, the current researches of optimization problem under the CEV model concern only one risky asset and a risk-free asset. But in most of real-world situations, an investor needs to invest in multiple risky assets. Thus, this paper deals with the investment problem with multiple risky assets under the CEV model. The investment objective is to maximize the expected utility of an investor's terminal wealth. By applying the method of stochastic optimal control, we derive a complicated non-linear

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partial differential equation (PDE). Owing to the difficulty of solution structure characterization, we employ a power transform and a variable change technique proposed by Cox (1996) to simplify the PDE. Since multiple risky assets are considered under the CEV model, closed-form solutions for the CARA utility function are obtained only for special cases (for the elasticity coefficient $\beta = -1$ and $\beta = 0$). In order to acquire a general optimal strategy for all values of the elasticity coefficient, we investigate a special investment problem with two risky assets and a risk-free asset (three-asset model). Suppose the price dynamics of risky assets conform to the CEV model and under a given assumption, a closed-form solution to the problem of expected CARA utility maximization is obtained. Each optimal strategy we derived under the CEV model in this paper contains two parts. The first part is akin to the optimal strategy under GBM model except in this part we have a volatility depending on the assets' prices. The second part can be explained as a modification factor resulted from the changes of volatilities. Furthermore, we present a numerical simulation to investigate the links between the two investment models. On one hand, we find that the effects of parameters on the optimal strategies and their modification factors under the two cases are similar. This indicates that the sensitivity analyses are influenced little by our assumption in the three-asset model. On the other hand, numerical results show that in the three-asset model, investors would like to take more risks than ones under general cases. This is due to the fact that the expected return of the optimal strategy is known in the three-asset model and thus investors will be more venturesome.

The paper proceeds as follows. In Section 2, we solve the portfolio selection problem under the CEV model and analyze the properties of the optimal strategy. Section 3 provides a special optimal investment problem with two risky assets and a risk-free asset, and gives the optimal strategy for all values of $\beta$, which enables us to discuss the effect of $\beta$ on the optimal investment. In Section 4, we provide a numerical analysis to demonstrate our results. Section 5 concludes the paper.

## 2. Portfolio selection problem with multiple risky assets

### 2.1. Formulation of the model

In this section, we consider a financial market consisting of one risk-free asset with price $S_0(t)$ given by

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \quad S_0(0) = 1 \tag{2.1}$$

and $n$ risky assets with prices $S_i(t)$ described by the CEV model

$$\mathrm{d}S_i(t) = S_i(t)\left(\mu_i \mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}(S_i(t))^{\beta}\mathrm{d}W_j(t)\right), \quad i = 1, 2, \ldots, n, \tag{2.2}$$

where $\mu_i$ is the appreciation rate of the $i$th risky asset and $r$ is the interest rate. $W := (W_1, \ldots, W_n)^{\mathrm{T}}$ is a $n$-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. $(\mathcal{F}_t)$ is an augmented filtration generated by the Brownian motion with $\mathcal{F} = \mathcal{F}_T$, where $T$ is a fixed and finite time horizon. Let $\mu := (\mu_1, \ldots, \mu_n)^{\mathrm{T}}$ be the appreciation rate vector. Define $\sigma = \{\sigma_{ij}\}_{n\times n}$ and

$$S^{\beta}(t) = \begin{pmatrix} (S_1(t))^{\beta} & 0 & \cdots & 0 \\ 0 & (S_2(t))^{\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (S_n(t))^{\beta} \end{pmatrix},$$

then $S^{\beta}(t)\cdot\sigma$ is the instantaneous volatility matrix. The elasticity parameter $\beta$ satisfies $\beta \le 0$. If $\beta = 0$, the volatility matrix is constant with respect to the prices of risky assets and (2.2) reduces to the Black–Scholes model. In addition, we assume that $\mu_i > r$ for each $i = 1, 2, \ldots, n$, and $\sigma\sigma^{\mathrm{T}}$ is positive definite throughout this paper.

The investor is allowed to invest in those $n$ risky assets as well as in the risk-free asset. Let $\pi_i(t)$ be the money amount invested in the $i$th risky asset at time $t$ for $i = 1, 2, \ldots, n$. Denote by $\pi(t) := (\pi_1(t), \ldots, \pi_n(t))^{\mathrm{T}}$ and each $(\pi_i(t))$ is an $(\mathcal{F}_t)$-predictable process for $i = 1, 2, \ldots, n$. Corresponding to a trading strategy $(\pi(t))$ and an initial capital $M$, the wealth process $(X(t))$ of the investor follows the dynamics

$$\mathrm{d}X(t) = [rX(t) + \pi^{\mathrm{T}}(t)(\mu - r\mathbf{1}_n)]\mathrm{d}t + \pi^{\mathrm{T}}(t)S^{\beta}(t)\sigma\mathrm{d}W(t), \quad X(0) = M, \tag{2.3}$$

where $\mathbf{1}_n = (1, \ldots, 1)^{\mathrm{T}}$ is an $n \times 1$ vector.

Suppose that the investor has a utility function $U$ which is strictly concave and continuously differentiable on $(-\infty, \infty)$. And he/she aims to maximize the expected utility of his/her terminal wealth, i.e.,

$$\max_{(\pi(t))} \mathrm{E}[U(X(T))]. \tag{2.4}$$

### 2.2. Solution to the model

By applying the classical tools of stochastic optimal control, we define the value function as

$$H(t, s_1, s_2, \ldots, s_n, x) = \sup_{(\pi(t))} \mathrm{E}\{U(X_T)|S_1(t) = s_1, S_2(t) = s_2, \ldots, S_n(t) = s_n, X(t) = x\}, \quad 0 < t < T \tag{2.5}$$

with $H(T, s_1, s_2, \ldots, s_n, x) = U(x)$.

The Hamilton–Jacobi–Bellman (HJB) equation associated with the portfolio selection problem under the CEV model is

$$H_t + \mu^{\mathrm{T}}SH_s + rxH_x + \frac{1}{2}\sum_{i=1}^{n} I_i^{\mathrm{T}}[S^{(\beta+1)}\sigma\sigma^{\mathrm{T}}S^{(\beta+1)}H_{ss}]I_i + \sup_{\pi}\left\{\pi^{\mathrm{T}}(\mu - r\mathbf{1}_n)H_x + \pi^{\mathrm{T}}S^{\beta}\sigma\sigma^{\mathrm{T}}S^{(\beta+1)}H_{xs} + \frac{1}{2}\pi^{\mathrm{T}}S^{\beta}\sigma\sigma^{\mathrm{T}}S^{\beta}\pi H_{xx}\right\} = 0, \tag{2.6}$$

where

$$S := \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{pmatrix}, \quad S^{\beta} := \begin{pmatrix} s_1^{\beta} & 0 & \cdots & 0 \\ 0 & s_2^{\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n^{\beta} \end{pmatrix},$$

$H_s := (H_{s_1}, \ldots, H_{s_n})^{\mathrm{T}}$, $H_{xs} := (H_{xs_1}, \ldots, H_{xs_n})^{\mathrm{T}}$ and

$$H_{ss} := \begin{pmatrix} H_{s_1s_1} & \cdots & H_{s_ns_1} \\ \vdots & \ddots & \vdots \\ H_{s_1s_n} & \cdots & H_{s_ns_n} \end{pmatrix}.$$

Besides, we define $I_i := (0, \ldots, 1, \ldots, 0)$, $i = 1, \ldots, n$, whose $i$th component is 1. Differentiating with respect to $\pi$ in (2.6) gives the optimal policy

$$\pi^* = -\left(S^{\beta}\sigma\sigma^{\mathrm{T}}S^{\beta}\right)^{-1}(\mu - r\mathbf{1}_n)\cdot\frac{H_x}{H_{xx}} - SH_{xs}\cdot\frac{1}{H_{xx}}. \tag{2.7}$$