Insurance: Mathematics and Economics 44 (2009) 337–344

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

# A jump-diffusion model for option pricing under fuzzy environments

Weidong Xu [a], Chongfeng Wu [a], Weijun Xu [b,*], Hongyi Li [c]

[a] Financial Engineering Research Center, Shanghai Jiaotong University, Shanghai 200052, China
[b] School of Business Administration, South China University of Technology, Guangzhou 510641, China
[c] Business Administration Faculty, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

ARTICLE INFO

*Article history:*
Received May 2008
Received in revised form July 2008
Accepted 10 September 2008

*JEL classification:*
G13
M01

*Keywords:*
Fuzzy numbers
Fuzzy random variable
European option pricing
Fuzzy jump-diffusion model

ABSTRACT

Owing to fluctuations in the financial markets from time to time, the rate $\lambda$ of Poisson process and jump sequence $\{V_i\}$ in the Merton's normal jump-diffusion model cannot be expected in a precise sense. Therefore, the fuzzy set theory proposed by Zadeh [Zadeh, L.A., 1965. Fuzzy sets. Inform. Control 8, 338–353] and the fuzzy random variable introduced by Kwakernaak [Kwakernaak, H., 1978. Fuzzy random variables I: Definitions and theorems. Inform. Sci. 15, 1–29] and Puri and Ralescu [Puri, M.L., Ralescu, D.A., 1986. Fuzzy random variables. J. Math. Anal. Appl. 114, 409–422] may be useful for modeling this kind of imprecise problem. In this paper, probability is applied to characterize the uncertainty as to whether jumps occur or not, and what the amplitudes are, while fuzziness is applied to characterize the uncertainty related to the exact number of jump times and the jump amplitudes, due to a lack of knowledge regarding financial markets. This paper presents a fuzzy normal jump-diffusion model for European option pricing, with uncertainty of both randomness and fuzziness in the jumps, which is a reasonable and a natural extension of the Merton [Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. J. Financ. Econ. 3, 125–144] normal jump-diffusion model. Based on the crisp weighted possibilistic mean values of the fuzzy variables in fuzzy normal jump-diffusion model, we also obtain the crisp weighted possibilistic mean normal jump-diffusion model. Numerical analysis shows that the fuzzy normal jump-diffusion model and the crisp weighted possibilistic mean normal jump-diffusion model proposed in this paper are reasonable, and can be taken as reference pricing tools for financial investors.



## 1. Introduction

There has been a lot of research work conducted to modify the Black–Scholes model (1973), based on Brownian motion and normal diffusion, in order to incorporate the two stylized empirical features: (1) The asymmetric leptokurtic features. (2) The volatility smile. More precisely, if the Black–Scholes model is correct, then the implied volatility calculated from the Black–Scholes model should be constant. However, it is widely recognized by empirical studies that the implied volatility is not a constant, but a convex curve of the strike price.

In order to incorporate the asymmetric leptokurtic features and the volatility smile into asset pricing, a variety of models has been proposed by modifying the Black–Scholes model based on Brownian motion and normal distribution. Merton (1976) was the first to consider a normal jump-diffusion model. This model can lead to the leptokurtic feature, implied volatility smile, and analytical solutions for the call and put options, and the interest rate derivatives. Heston (1993) and Duffie and Singleton (2000) proposed affine stochastic volatility and affine jump-diffusion models. Fractional Brownian motion has been used by Rogers (1997) to model the movement of log asset prices, which would allow long-range dependence between returns on different days. Samorodnitsky and Taqqu (1994) used stable distribution to model the distribution of asset prices. Carr and Wu (2004) proposed a time-changed Lévy process to study option pricing. A double exponential jump-diffusion model was proposed by Kou (2002), Kou and Wang (2004) which can lead to an analytic approximation for the finite-horizon American options and analytical solutions for the popular path-dependent options (such as the lookback, barrier, and perpetual American options). The main difference between the double exponential jump-diffusion model and the normal jump-diffusion model is the analytical tractability for the path-dependent options. Further, there are many models proposed from chaos theory, fractal Brownian motion, and stable processes.

In a parallel development, the fuzzy set theory is applied to model option pricing. Simonelli (2001) provided a methodology

* Corresponding address: School of Business Administration, South China University of Technology, No. 381, Wushan Road, Tianhe District, 510641, Guangzhou City, Guangdong Province, China. Tel.: +86 20 32745925.

*E-mail addresses:* xuweidong@sjtu.edu.cn (W. Xu), cfwu@sjtu.edu.cn (C. Wu), xuwj@scut.edu.cn (W. Xu), hongyi@baf.msmail.cuhk.edu.hk (H. Li).

0167-6687/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2008.09.003

evaluating financial instruments using certainty equivalents. However, the approach by Simonelli was not based on the Black–Scholes formula. Andrés and Terceño (2004) estimated a fuzzy term structure of interest rates using fuzzy regression techniques. Yoshida (2003) and Yoshida et al. (2006) discussed the valuation of European and American options with uncertainty for both randomness and fuzziness in the output variables, by introducing fuzzy logic to the stochastic financial model. Lee et al. (2005) adopted the fuzzy decision theory and Bayes' rule for measuring fuzziness in the practice of option analysis. Wu (2004, 2005, 2007) obtained the so called fuzzy pattern of the Black–Scholes formula in his papers when the arithmetics in the (conventional) Black–Scholes formula are replaced by fuzzy arithmetic. Muzzioli and Torricelli (2004) studied the multiperiod binomial model for pricing options in a fuzzy world. Chrysafis and Papadopoulos (2007)presented an application of a new method of constructing a fuzzy estimator for volatility in Black–Scholes formula, and also analyzed the results for the Greek parameters. The book of collected papers edited by Ribeiro et al. (1999) gave the applications of fuzzy set theory to the discipline called financial engineering.

The fuzzy set concept was introduced by Zadeh (1965). For a real number $a$, the fuzzy number $\tilde{a}$ corresponding to $a$ can be interpreted as "around $a$". The fuzzy number $\tilde{a}$ is defined by its membership function $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{a}}(a) = 1$. It means that the membership degree $\mu_{\tilde{a}}(x)$ is close to 1 when the value $x$ is close to $a$. The concept of fuzzy random variable was introduced by Kwakernaak (1978) and Puri and Ralescu (1986). An excellent overview of the fuzzy random variable can be found in Shapiro (2008). The development of fuzzy random variables enables the joint effort of randomness and fuzzy set theory to better model imprecision. Dubois and Prade (1987, 1988) introduced the mean value of a fuzzy number as a closed interval, bounded by the expectations calculated from its upper and lower distribution functions. Carlsson and Fullér (2001) and Fullér and Majlender (2003) introduced the possibilistic and weighted possibilistic mean values of a fuzzy number, respectively.

In Merton's normal jump-diffusion model, the rate $\lambda$ of a Poisson process and the jump sequence $V_i$ are assumed as precise real numbers. However, in the real financial market, the average jump times and jump amplitudes of asset prices cannot always be expected to be constant over time. Therefore, the fuzzy set theory proposed by Zadeh (1965) and the fuzzy random variable introduced by Kwakernaak (1978) and Puri and Ralescu (1986) may be useful for modeling this kind of imprecise problem. In this paper, probability is applied to characterize the uncertainty as to whether jumps occur or not, and what the amplitudes of the jumps are, while fuzziness is applied to describe the uncertainty of not being able to specify the exact values of jump times and jump amplitudes, due to lack of knowledge of the financial market. This paper presents a fuzzy normal jump-diffusion model for the European option pricing, with uncertainty of both randomness and fuzziness in jumps, which is a reasonable and natural extension of the Merton (1976) normal jump-diffusion model. Based on the crisp weighted possibilistic mean values of fuzzy variables in fuzzy normal jump-diffusion model, we also obtain the crisp weighted possibilistic mean normal jump-diffusion model.

This paper is organized as follows. In Section 2, the notations of fuzzy numbers, weighted possibilistic mean values of fuzzy numbers, and fuzzy random variables are introduced. In Section 3, the analytical solutions for an European option are obtained under fuzzy normal jump-diffusion model and the crisp weighted possibilistic mean values of fuzzy variables in fuzzy normal jump-diffusion model. In Section 4, a numerical analysis is performed to obtain the European option price under the crisp weighted possibilistic mean normal jump-diffusion model, and Merton's normal jump-diffusion model. Finally, the conclusions of this paper are stated in Section 5.

## 2. Fuzzy set theory

### 2.1. Fuzzy numbers and extension principle

Let $\mathcal{R}$ be the set of all real numbers. Then a fuzzy subset $\tilde{A}$ of $\mathcal{R}$ is defined by its membership function $\mu_{\tilde{A}} : \mathcal{R} \to [0, 1]$. The $\alpha$-level set of $\widetilde{A}$ is defined by $\widetilde{A}_\alpha = \{x|\mu_{\widetilde{A}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. The 0-level set $\widetilde{A}_0$ of $\tilde{A}$, is defined by the closure the set $\{x|\mu_{\widetilde{A}}(x) > \alpha\}$. Now, $\tilde{A}$ is called normal fuzzy set if there exists an $x$, such that $\mu_{\widetilde{A}}(x) = 1$, and $\widetilde{A}$ is called a convex fuzzy set if $\mu_{\widetilde{A}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\}$ for $\forall \lambda \in [0, 1]$; that is, $\mu_{\widetilde{A}}$ is a quasi-concave function.

Throughout this paper, the universal set $\mathcal{R}$ is assumed to be the set of all real numbers endowed with a usual topology. Let $f$ be a real-valued function defined on $\mathcal{R}$. Then $f$ is said to be upper semicontinuous if $\{x|f(x) \geq \alpha\}$ is a closed set for each $\alpha$. A fuzzy number is a fuzzy subset defined over a real number. The fuzzy number $\tilde{a}$ corresponding to $a$ can be interpreted as "around" $a$ and $\mu_{\tilde{a}}(a) = 1$.

**Definition 2.1** (*Wu, 2004*). Let $\widetilde{a}$ be a fuzzy subset of $\mathcal{R}$. Then $\widetilde{a}$ is called a fuzzy number if the following three conditions are satisfied:

(i) $\tilde{a}$ is a normal and convex fuzzy set;
(ii) Its membership function $\mu_{\widetilde{A}}(x)$ is upper semi-continuous;
(iii) The $\alpha$-level set $\tilde{a}_\alpha$ is bounded for each $\alpha \in [0, 1]$.

From Zadeh (1965), $\widetilde{A}$ is a convex fuzzy set if and only if, its $\alpha$-level set $\widetilde{A}_\alpha = \{x|\mu_{\widetilde{A}}(x) \geq \alpha\}$ is a convex set for all $\alpha \in [0, 1]$. Therefore, if $\tilde{a}$ is a fuzzy number, then the $\alpha$-level set $\tilde{a}_\alpha$ is a compact (closed and bounded in $\mathcal{R}$) and convex set; that is, $\tilde{a}$ is a closed interval in $\mathcal{R}$. The $\alpha$-level set of $\tilde{a}$ is then denoted as by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$.

**Proposition 2.1** (*Resolution identity, (Zadeh, 1975)*). *Let $\widetilde{A}$ be a fuzzy set with membership function $\mu_{\widetilde{A}}$ and $\widetilde{A}_\alpha = \{x|\mu_{\widetilde{A}}(x) \geq \alpha\}$. Then*

$$\mu_{\widetilde{A}}(x) = \sup_{\alpha\in[0,1]} \alpha[1_{\widetilde{A}_\alpha}(x)]$$

*where $1_{\widetilde{A}_\alpha}$ is an indicator function of set $\widetilde{A}_\alpha$, i.e., $1_{\widetilde{A}_\alpha}(x) = 1$ if $x \in \widetilde{A}_\alpha$ and $1_{\widetilde{A}_\alpha}(x) = 0$ if $x \notin \widetilde{A}_\alpha$. Note that the $\alpha$ set $\widetilde{A}_\alpha$ of $\widetilde{A}$ is a crisp (usual) set.*

Note that $\tilde{a}$ is called a crisp number with value $a$, if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

which is denoted by $\tilde{a} = \tilde{1}_{\{a\}}$. It is easy to see that $(\tilde{1}_{\{a\}})^L_\alpha = (\tilde{1}_{\{a\}})^U_\alpha = a$ for all $\alpha \in [0, 1]$. We see that real numbers are a special case of fuzzy numbers when real numbers are regarded as crisp numbers.

Kaufmann and Gupta (1985) discussed the arithmetics of any two fuzzy numbers. Let "$\odot$" be a binary operator $\oplus$, $\ominus$, $\otimes$, or $\oslash$ between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\mu_{\tilde{a}\odot\tilde{b}} = \sup_{\{(x,y)|x\circ y=z\}} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\},$$

where the binary operator $\oplus$, $\ominus$, $\otimes$, or $\oslash$ correspond to the binary operator $\circ = +, -, \times$, or / according to the "Extension Principle" in Zadeh (1975). Zadeh's extension principle is often referred to in the fuzzy literature as the sup min extension principle. This