Insurance: Mathematics and Economics 48 (2011) 304–313

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

# An operator-based approach to the analysis of ruin-related quantities in jump diffusion risk models

Runhuan Feng *

*Department of Mathematical Sciences, University of Wisconsin - Milwaukee, P.O. Box 413, Milwaukee, WI, 53202-0413, USA*

A R T I C L E   I N F O

*Article history:*
Received August 2010
Received in revised form December 2010
Accepted 10 December 2010

*Keywords:*
Jump diffusion process
Ruin theory
Expected discounted penalty at ruin
Integro-differential equation
Operator calculus
Resolvent density

A B S T R A C T

Recent developments in ruin theory have seen the growing popularity of jump diffusion processes in modeling an insurer's assets and liabilities. Despite the variations of technique, the analysis of ruin-related quantities mostly relies on solutions to certain differential equations. In this paper, we propose in the context of Lévy-type jump diffusion risk models a solution method to a general class of ruin-related quantities. Then we present a novel operator-based approach to solving a particular type of integro-differential equations. Explicit expressions for resolvent densities for jump diffusion processes killed on exit below zero are obtained as by-products of this work.



## 1. Introduction

Historically, most insurance-related problems in ruin theory are natural applications of jump processes due to the nature of insurance claims which occur at discrete time points, whereas many classical models in financial mathematics rely on continuous processes to reflect fluctuations in the constantly changing financial markets. Although the two disciplines of applied probability have evolved rather independently, there is a common trend in recent years to incorporate stochastic models with both continuous and jump components. For example, on the ruin theory side, in addition to the random jumps which account for insurance claims, diffusion components have gained increasing popularity to describe investment returns in sophisticated risk models. On the ground of previous works in both areas, we shall investigate a rather general jump diffusion model in which the analysis of ruin-related quantities is the main objective and can be extended for other quantities of interests.

To give a motivation of the jump diffusion risk model under consideration, we first review the basic structure of classical risk models. Despite various forms in the literature, a risk model typically consists of assumptions on two offsetting cashflows. (1) The incoming cashflow is usually generated by premium income, investment returns, capital injections, etc. (2) The outgoing cashflow is composed of insurance claims paid to policy holders, dividends paid to shareholders and business overhead costs, etc.

Given the basic structure of risk models, jump diffusion processes are well suited for modeling purposes. On the asset side, the drift component can be chosen to reflect the dominating trend of growth in surplus and the volatility component accounts for the randomness in surplus due to unforeseeable events, whereas on the liability side, the jump component represents an insurer's claim payments or losses due to extreme events on a large scale. To allow for more flexibility, one could model the actual jump in surplus caused by a random claim, by allowing its dependency on both the size of the claim and the surplus level prior to the claim.

Translating the descriptions above into probabilistic terms, we assume an insurer's surplus is modeled by $X = \{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ together with a family of probability measures $\{\mathbb{P}^x, x \in \mathbb{R}\}$ satisfying the usual conditions. The surplus is determined by the following stochastic differential equation (SDE)

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t - a(X_{t-})\,\mathrm{d}Z_t, \tag{1.1}$$

with $\mathbb{P}^x(X_0 = x) = 1$, where $a(\cdot)$ measures the actual impact of a claim on the surplus level, $B$ is an adapted Brownian motion and $Z$ is an adapted pure jump process. Certain integrability conditions are required for $\mu$, $\sigma$ and $a$ to ensure the existence and pathwise uniqueness of the process. Many well-known stochastic models in ruin literature can be characterized by this process. Among many others, examples include various compound Poisson risk models in Asmussen (2000) and Gerber (1979), pure diffusion risk models in Gerber and Shiu (2006) and Cai et al. (2006) as well as risk

* Tel.: +1 4142296109; fax: +1 4142294907.
*E-mail address:* fengr@uwm.edu.

0167-6687/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2010.12.003

models involving stochastic investment returns such as Højgaard and Taksar (2001), Jang (2007) and Yuen et al. (2007), etc.

Despite its generality, the model (1.1) has its own limitations for more general applications. For instance, if the size of increase in the aggregate claim is $\Delta Z(t) = z$, then the surplus will have a jump due to the impact measured by $\Delta X(t) = a(X_{t-})z$. However, for practical applications, the jump in surplus could be dependent on both $X(t-)$ and $z$, but not necessarily linear in $z$. It says in reality that a relatively large insurance claim on a low surplus causes far more financial stress than a small claim on a more sustainable high surplus. With the advances in Lévy processes, such generality can be easily tackled by using a Lévy-type stochastic integral.

From now on, we shall build our analysis on a process defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and which is more general than (1.1). To make the process as inclusive as possible in the context of ruin theory, we define the real-valued surplus process $X = (X_t, t \geq 0)$ by the following stochastic integral,

$$\mathrm{d}X_t = \mu(X_{t-})\,\mathrm{d}t + \boldsymbol{\sigma}(X_{t-})^\top \mathrm{d}\mathbf{B}_t + \int_{-\infty}^{\infty} \mathbf{F}(X_{t-}, z)^\top \overline{\mathbf{N}}(\mathrm{d}t,\ \mathrm{d}z), \tag{1.2}$$

where $\boldsymbol{\sigma}^\top = (\sigma_1, \ldots, \sigma_m) : \mathbb{R} \mapsto \mathbb{R}^m, \mathbf{F}^\top = (F_1, \ldots, F_p) : \mathbb{R} \times \mathbb{R}^p \mapsto \mathbb{R}^p, \mathbf{B} = (B_1, \ldots, B_m)$ is an $m$-dimensional standard Brownian motion and $\overline{\mathbf{N}}(\mathrm{d}t, \mathrm{d}z)^\top = (N_1(\mathrm{d}t, \mathrm{d}z_1) - \nu_1(\mathrm{d}z_1)\mathrm{d}t, \ldots, N_p(\mathrm{d}t, \mathrm{d}z_p) - \nu_p(\mathrm{d}z_p)\mathrm{d}t)$, and $\{N_j,\ j = 1, \ldots, p\}$ are independent one-dimensional Poisson random measures with Lévy measures $\{\nu_j,\ j = 1, \ldots, p\}$ such that $\nu_j(A) = \mathbb{E}N_j(1, A)$ for any Borel set $A \subset \mathbb{R}$ such that $0 \notin \overline{A}$. We introduce the notation $a(x, y) = \boldsymbol{\sigma}(x)^\top \boldsymbol{\sigma}(y)$. Readers are referred to Applebaum (2004) and Øksendal and Sulem (2007) for detailed accounts of Lévy-type stochastic integrals. It is known (*c.f.* Section 6.2 of Applebaum, 2004) that if there exist constants $K_1$ and $K_2$ such that for all $x_1, x_2 \in \mathbb{R}$

$$|\mu(x_1) - \mu(x_2)|^2 + |a(x_1, x_1) - 2a(x_1, x_2) + a(x_2, x_2)| + \sum_{k=1}^{p} \int_{-\infty}^{\infty} |F_k(x_1, z) - F_k(x_2, z)|^2 \nu_k(\mathrm{d}z) \leq K_1 |x_1 - x_2|^2,$$

and for all $x \in \mathbb{R}$,

$$|\mu(x)|^2 + |a(x, x)| + \sum_{k=1}^{p} \int_{-\infty}^{\infty} |F_k(x, z)|^2 \nu_k(\mathrm{d}z) \leq K_2(1 + |x|^2), \tag{1.3}$$

then the stochastic process $X$ exists and is pathwise unique. For practical applications, it is reasonable to assume that $\nu$ is a finite measure. Hence we can show by Itô's formula for semimartingales that the infinitesimal generator of $X$ given in (1.2) is given by

$$\mathcal{A}f(x) = \hat{\mu}(x)f'(x) + \frac{1}{2}a(x, x)f''(x) + \sum_{k=1}^{p} \int_{-\infty}^{\infty} \{f(x + F_k(x, z)) - f(x)\}\nu_k(\mathrm{d}z), \tag{1.4}$$

where

$$\hat{\mu}(x) = \mu(x) - \sum_{k=1}^{p} \int_{-\infty}^{\infty} F_k(x, z)\nu_k(\mathrm{d}z).$$

The focus of this paper is to present a solution method to the following quantity, which shall be called the expected present value of total operating costs up to default,

$$H(x) = \mathbb{E}^x \left[ \int_0^{\tau_d} \mathrm{e}^{-\delta t} l(X_t)\,\mathrm{d}t \right], \tag{1.5}$$

where $\delta \geq 0$, the event of default is defined by $\tau_d = \inf\{t | X_t < d\}$ with the convention that $\inf \varnothing = \infty$, the $\mathcal{B}(\mathbb{R})$-measurable function $l$ represents the operating cost depending on the surplus level and $d$ is a prescribed level of default. When $d = 0$, the event of default reduces to what is commonly known as the event of bankruptcy in ruin theory. This quantity (1.5) is known as a solution to the Poisson equation and is well-studied in the context of diffusion processes. An overview of the Poisson equation can be found in Bass (1997). There are also extensive research works on such quantity in the context of Lévy-type stochastic integrals for stochastic control problems, *c.f.* Øksendal and Sulem (2007).

The quantity (1.5) is first introduced in ruin literature by Cai et al. (2009) in the context of piecewise-deterministic compound Poisson process and can be viewed as a generalization of the Gerber–Shiu expected discounted penalty function. The relationship between (1.5) and various other ruin-related quantities has been explored in more general renewal risk models such as Feng (2009a,b). It is remarkable to note that in all previously mentioned risk models the quantity (1.5) is shown to include the Gerber–Shiu function and expected present value of dividends under various dividend strategies, as well as many other quantities. The generality of this quantity enables us to focus on an "all-in-one" solution method to all quantities in its family, rather than putting redundant efforts to compute them individually. It also has the technical advantage of providing a framework to extend the analysis of ruin to that of default, which is of evident importance in the study of credit risk in both financial and insurance industry. Readers may refer to Crouhy et al. (2000) for a detailed account of distinction between default and bankruptcy and their significance from the standpoint of risk management.

This paper attempts to address the solution method of the quantity (1.5) in the following two steps. (1) We demonstrate in Section 2 that the solution indeed satisfies a Poisson equation under certain smoothness conditions. Although experienced researchers may view this as a "folklore result", we provide a rigorous proof for lack of direct references in the literature on this particular result and also in order to justify the smoothness conditions which may not seem obvious. Two well-known examples are supplied to illustrate a quick path to intermediate differential equations which would otherwise require onerous work. (2) This paper also presents a novel operator-based approach to solve the Poisson equation in the context of a jump-diffusion model in Section 3, followed by numerous examples of ruin-related quantities with explicit solutions. This operator-based approach appeared in various forms in the context of different risk models such as Cai et al. (2009) and Feng (2009b). In a forthcoming paper by Feng and Shimizu (2010), the operator-based approach is further extended to analyze a more general Lévy risk model. It should be pointed out that a subset of the operator identities shown in Appendix B has been used in Albrecher et al. (2010) as a basis for automated computer algebra systems, such as Mathematica, for solving boundary value problems in the context of a renewal risk model. The approach in the present paper continues in this direction and shows the potential of further broadening the spectrum of problems solvable by computer algebra systems.

Another interesting by-product of this work is to derive the so-called *resolvent density* $R_\delta(x, y)$, which is defined by

$$\mathbb{E}^x \left[ \int_0^\infty \mathrm{e}^{-\delta s} f(X_s)\,\mathrm{d}s \right] = \int_{\mathbb{R}} R_\delta(x, y) f(y)\,\mathrm{d}y,$$

for all nonnegative measurable $f$ on $\mathbb{R}$. To the best knowledge of the author, there is scarce literature on resolvent densities for jump diffusion processes. We produce in Section 3 an explicit expression of the resolvent density for a superposition of a compound Poisson process and a Brownian motion killed on exiting $[0, \infty)$. The