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Continuous-time portfolio selection with liability: Mean–variance model and stochastic LQ approach[☆]

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Abstract

In this paper we formulate a continuous-time mean–variance portfolio selection model with multiple risky assets and one liability in an incomplete market. The risky assets' prices are governed by geometric Brownian motions while the liability evolves according to a Brownian motion with drift. The correlations between the risky assets and the liability are considered. The objective is to maximize the expected terminal wealth while minimizing the variance of the terminal wealth. We derive explicitly the optimal dynamic strategy and the mean–variance efficient frontier in closed forms by using the general stochastic linear-quadratic (LQ) control technique. Several special cases are discussed and a numerical example is also given.

Keywords: Portfolio selection; Asset–liability management; Continuous-time; Mean–variance model; Stochastic linear-quadratic control

1. Introduction

Since the pioneer work of Markowitz (1952) the mean–variance (M–V) portfolio selection model has inspired literally hundreds of extensions and applications. For example, Merton (1971) studied a continuous-time model with consumption; Koo (1998) took labor income into account; Li and Ng (2000) studied a dynamic multi-period M–V problem; Zhou and Li (2000) investigated a continuous-time M–V portfolio problem in a stochastic LQ framework; Zhu et al. (2004) proposed a dynamic multi-period M–V portfolio selection model with bankruptcy prohibition, while Bielecki et al. (2005) sought the same problem in a continuous-time setting. For more detailed discussions on this subject, one can refer to Li and Wang (2001) and Wang and Xia (2002).

The above mentioned authors did not incorporate liability into their models. However, in most of the real-world situations, liability is an important factor which almost all investors should cope with. It is clear that the introduction of liability in a portfolio selection model will make it more practical.

The existing literature on liability focused primarily on market models of firm's asset–liability management (ALM), optimal dividend pay-out and ruin problems; see, for example, Sharpe and Tint (1990), Keel and Muller (1995), Norberg (1999), Gerber and Shiu (2004), and Decamps et al. (2006). However, the research on dynamic M–V portfolio selection with liability is limited, and it evokes recent concern. Leippold et al. (2004) studied a multi-period asset–liability management problem under the M–V criteria, and derived explicit expressions for the optimal strategy and the M–V efficient frontier by using a geometric approach and the embedding technique of Li and Ng (2000). Chiu and Li (2006)

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investigated a continuous-time asset–liability management problem under the M–V criteria and in the setting in which both the risky assets' prices and the liability are assumed to be governed by the same Brownian motions, and provided analytical formulae for the optimal policy and the M–V efficient frontier by employing the stochastic control theory.

In this paper, we investigate the problem of continuous-time portfolio selection with liability in an incomplete financial market. In the market, m risky assets and one risk-free asset are traded continuously, the risky assets' dynamic prices are driven by an n-dimensional Brownian motion with $n \geq m$, the liability is dynamically exogenous and its driving factors include but do not equal to the ones of the risky assets' prices. There exist correlations between the liability and the risky assets' prices. As a special case, when the sum of these correlation coefficients' squares is equal to one, both the risky assets' prices and the liability are driven by the same Brownian motions as considered by Chiu and Li (2006).

Furthermore, we assume that the prices of the risky assets evolve according to geometric Brownian motions while the liability is governed by a Brownian motion with drift. The representation of a liability as a Brownian motion with drift can be found, for example, in Norberg (1999) and Decamps et al. (2006). Additionally, in Henderson and Hobson (2004), the stock price was described as a geometric Brownian motion and the auxiliary non-traded stock's price was submitted to a Brownian motion with drift. The non-traded stock discussed in that paper, just like the liability in this paper, is dynamically exogenous and is related to the price evolvement of the traded risky asset.

It is noteworthy that Zhou and Li (2000) employed the stochastic linear quadratic (LQ) technique to tackle a continuous-time pure investment problem under the complete market assumption, i.e., $n = m$. The diffusion terms in the wealth differential equation derived by Zhou and Li (2000) are independent and identically distributed and their diffusion coefficients are homogeneously linear with respect to the control variables. So the optimal portfolio strategy and the M–V efficient frontier for pure investment case can be derived by solving an auxiliary typical stochastic LQ control problem.

We emphasize, however, that the introduction of a liability is by no means routine and does give rise to difficulties which are not encountered in the pure investment case of Zhou and Li (2000).

Firstly, due to the correlations between the risky assets and the liability, the wealth differential equation derived from our model cannot agree with the representation of the state equation in the general framework of stochastic LQ control, while the latter requires the independent property of the diffusion terms. Enlightened from Koo (1998), we overcome this difficulty by rewriting the diffusion term of the liability.

Secondly, since the diffusion coefficients in the wealth differential equation derived from our model are non-homogeneous with respect to the control variables, the stochastic LQ control framework stated in the pure investment case of Zhou and Li (2000) is no longer valid for our model. By applying the more general stochastic LQ control technique in Yong and Zhou (1999) to our model, we introduce a stochastic LQ auxiliary control problem and derive its optimal feedback control. Eventually the optimal portfolio strategy and the efficient frontier for the original M–V portfolio optimization problem with a liability are obtained in closed forms.

The paper proceeds as follows. In Section 2 we model the continuous-time M–V portfolio optimization problem with a liability. In Section 3 we introduce an auxiliary problem which is transformed into a stochastic LQ control problem. In Section 4 we derive the optimal feedback control of the auxiliary problem. Section 5 is devoted to derive explicit expressions of the optimal strategy and the M–V efficient frontier for the problem formulated in Section 2. In Section 6 we discuss several special cases of our results. In Section 7 we demonstrate a numerical example to show the effect of the liability and the incompleteness in the market. Section 8 concludes the paper.

2. The model

Let $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{F})$ be a complete filtered probability space. Assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is generated by an $(n+1)$-dimensional standard Brownian motion $\{(W_t^0, W_t^1, \ldots, W_t^n)' : t \in [0, T]\}$ for a positive integer n, where the positive number T is a fixed and finite time horizon, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, and the superscript "′" represents the transpose of a vector or a matrix.

We denote by $\mathcal{C}([0, T]; \mathbb{R}^{n\times k})$ the class of $\mathbb{R}^{n\times k}$-valued continuous bounded deterministic functions on $[0, T]$, and by $\mathcal{L}^2_{\mathcal{F}}([0, T]; \mathbb{R}^m)$ the class of all $\mathbb{R}^m$-valued, progressively measurable and square integral random variables on $[0, T]$ under $\mathcal{P}$ with norm

$$\|\xi\|_{\mathcal{L}^2_{\mathcal{F}}} := \left(E\int_0^T |\xi_t|^2 \mathrm{d}t\right)^{\frac{1}{2}} < \infty, \quad \forall \xi_t \in \mathcal{L}^2_{\mathcal{F}}([0, T]; \mathbb{R}^m).$$

Consider an investor equipped with an initial endowment $w > 0$ and an initial liability l ($l \in \mathbb{R}$) at time $t = 0$. Denote by x the net initial wealth of the investor, i.e., $x = w - l$. The investor is allowed to adjust his/her portfolio during the time interval $[0, T]$, and short-selling is allowable.

We consider a financial market in which $(m + 1)$ assets are traded continuously within the time horizon $[0, T]$. We label these assets by $i = 0, 1, \ldots, m$, where $m \leq n$ and the 0th asset represents the risk-free asset. The risk-free asset's price A_t^0 is subject to the following (deterministic) ordinary differential equation (ODE):

$$\mathrm{d}A_t^0 = r_t A_t^0 \mathrm{d}t, \quad A_0^0 = 1, \tag{1}$$

where $r_t \in \mathcal{C}([0, T]; \mathbb{R}^+)$ is the interest rate of the risk-free asset. The remaining m assets are risky and their price processes $A_t^1, A_t^2, \ldots, A_t^m$ satisfy the following stochastic differential equations (SDEs):

$$\mathrm{d}A_t^i = \mu_t^i A_t^i \mathrm{d}t + \sigma_t^i A_t^i \mathrm{d}W_t, \quad A_0^i = a_i \in \mathbb{R}, \quad i = 1, 2, \ldots, m, \tag{2}$$