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Worst case risk measurement: Back to the future?☆

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ABSTRACT

This paper studies the problem of finding best-possible upper bounds on a rich class of risk measures, expressible as integrals with respect to measures, under incomplete probabilistic information. Both univariate and multivariate risk measurement problems are considered. The extremal probability distributions, generating the worst case scenarios, are also identified.

The problem of worst case risk measurement has been studied extensively by Etienne De Vijlder and his co-authors, within the framework of finite-dimensional convex analysis. This paper revisits and extends some of their results.

1. Introduction

This paper studies the problem of finding best-possible upper bounds on risk measures when there is incomplete probabilistic information on the risks under consideration. Traditionally, the measurement of *univariate* risk under incomplete probabilistic information has been a main problem in non-life insurance. More recently, the measurement of *multivariate* risk under incomplete probabilistic information has become increasingly important. Developments in the insurance and financial industry, such as the (current and upcoming) solvency capital accords and the explosive growth in multi-name financial derivative products, urge for appropriate techniques for the measurement of multivariate risk (Genest et al., 2009).

A main problem for the measurement of risk is that in practice often (only) partial information is available on the risks under study. A sound approach in that case is to investigate conditional worst case scenarios: given characteristics that only partially describe the risks, one searches for the most adverse scenario. This is also a good strategy for stress testing. For example, in a multivariate setting with a lack of information on the level of dependence between several risks, one could assume marginal distributions to be known, and identify the most adverse dependence scenario, conducting a stress test for dependence.

In this paper, we will mainly be concerned with a rich class of risk measures, namely risk measures that are expressible as integrals with respect to measures. The availability of partial probabilistic information is formalized by imposing integral constraints, and conical constraints on the cone of measures under

☆ In honor of Etienne De Vijlder. Etienne passed away in 2004. To honor his large contribution to various fields within Actuarial Science – not limited to the type of problems discussed in this paper – we dedicate this work to him.

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consideration. We study both univariate and multivariate risk measurement problems in this setting.

In the financial mathematics literature, the elements of (possibly degenerate) sets of measures are sometimes called *generalized scenarios*, and the worst case measure generating the best-possible upper bound on a risk measure is often referred to as the *worst case scenario*. In general, according to Wald (1950), worst case risk measurement, that is, using a maxmin decision criterion, seems reasonable when an a priori probability measure does not exist or is unknown to the decision maker; see also Huber (1981) and Gilboa and Schmeidler (1989) for references in this direction. For policy making purposes, worst case risk measurement should not be viewed as a substitute for Bayesian decision making à la Savage (1954), but rather as a way of constructing and assessing a prior; see also the discussion in Sims (2001).

The problem of finding best-possible upper bounds on integrals with respect to measures has been studied extensively by the late Etienne De Vijlder and his co-authors; see De Vylder (1982, 1983a,b,c, 1996),[1] De Vylder and Goovaerts (1982, 1983a,b) and Goovaerts et al. (1982, 1984). They considered the problem within the framework of finite-dimensional convex analysis. A main topic in convex analysis, and in convex optimization in particular, is to find the extremes of a (quasi-) convex function on a finite-dimensional convex body. The interested reader is referred to Ekeland and Temam (1976), Ioffe and Tikhomirov (1979), Tikhomirov (1996) and Borwein and Lewis (2000) for further details on finite-dimensional convex analysis.

A different approach is taken for the *reduced problem of moments*, which was studied already by Markov (1884) and Riesz (1911) in the late nineteenth and early twentieth century. Here, n moments $\mu_1, \mu_2, \dots, \mu_n$ are given, and one tries to find extremal values of the expected value of a certain function of the risk having these moments. An example is the stop-loss premium $\mathbb{E}[(X - t)_+]$ for a certain fixed t, or simply the tail probability $\mathbb{P}[X > t]$. The existence of a feasible distribution satisfying these moment constraints can be expressed by means of a quadratic form that has to be non-negative; see, e.g., Theorem 2.1.3 in Kaas (1987). In case the moment problem allows a solution, it can be calculated by the construction of polynomials that are upper (or lower) bounds for the function involved on the interval considered. Choosing the optimal polynomials boils down to finding polynomials that are tangent to the function considered. The points of intersection of the polynomials and the function considered are just the support of the extremal distribution. For different intervals in which the support must lie, e.g., $[0, +\infty)$, different solutions are obtained.[2] The integral constraints here are restricted to be moment constraints of type $\mathbb{E}[X^k] = \mu_k$. An early reference is Shohat and Tamarkin (1943); see also Isii (1960, 1963) and Kemperman (1968). Generalizations of this scheme are the Tchebycheff systems; see Karlin and Studden (1966). The general problem studied in this paper is also related to mass transportation problems (MTP); see Rachev and Rüschendorf (1998) for a detailed account, and also Sections 5 and 6.

This paper revisits and extends some of the work on best-possible upper bounds for integrals with respect to measures, done by Etienne De Vijlder and his co-authors. The extension is twofold: (i) we study some topical univariate risk measurement problems; and (ii) we demonstrate that the general theory considered by Etienne De Vijlder and his co-authors to a large extent allows application to worst case measurement of multivariate risk, while only applications to problems of univariate risk have been considered hitherto.

Other contributions to the problem of worst case measurement of univariate risk with applications to insurance and finance in mind include among others, Taylor (1977), Goovaerts and Kaas (1985), Brockett and Cox (1985) and Denuit et al. (1999b).

The problem of finding best-possible upper bounds on measures of multivariate risk when the marginal distributions are known, has a rich history in probability theory, where it typically appears under the name *Fréchet problem*. The problem of bounding the distribution function of a sum of random variables with given marginal distributions can be traced back to A.N. Kolmogorov. It was solved by Makarov (1981) in a two-dimensional setting, and, using different routes, by Rüschendorf (1982, dual approach) and Frank et al. (1987, copula-based approach). Recent contributions to the problem of multivariate risk measurement under incomplete information with given marginals include Denuit et al. (1999a, 2005), Kaas et al. (2000), Dhaene et al. (2002), Rüschendorf (2004), Embrechts et al. (2005), Embrechts and Puccetti (2006), Kaas et al. (2009) and Laeven (2009).

The rest of this paper is organized as follows: Section 2 presents the general problem, and recalls and discusses some main results. In Section 3, we introduce the *special dual problem* to which the general problem can often be reduced. In Section 4, we consider a discretization method to solve the special dual problem numerically. Section 5 discusses some strengths and shortcomings of the dual theory presented. In Section 6, we illustrate the dual approach with some univariate as well as multivariate examples. Finally, Section 7 contains some concluding remarks.

2. The general problem

We consider a measurable space $(\Omega, \mathcal{A})$ and $\mathcal{A}$-measurable functions $f, g_i : \Omega \to \mathbb{R}$, $i = 1, \dots, n$. Furthermore, we consider a cone $\mathcal{M}$ of countably additive measures $\mu : \mathcal{A} \to \mathbb{R} \cup \{-\infty, +\infty\}$. The cone $\mathcal{M}$ need not be pointed, i.e., it does not necessarily contain the zero measure. We note that a cone need not be a convex set; in many applications, however, the cone $\mathcal{M}$ will contain non-negative measures only, in which case $\mathcal{M}$ is a true convex set. All integrals in this section are over Ω.

We consider the following problem:

$$v(\mathbf{c}) = \sup_{\mu \in \mathcal{M}} \left(\int f \mathrm{d}\mu \mid \int g_i \mathrm{d}\mu = c_i,\ i = 1, \dots, n \right), \tag{1}$$

with the real-valued vector $\mathbf{c} = (c_1, \dots, c_n)$ and the functions $f, g_i,\ i = 1, \dots, n$, fixed and given. We assume that $\int f \mathrm{d}\mu$ and $\int g_i \mathrm{d}\mu,\ i = 1, \dots, n$, exist for all $\mu \in \mathcal{M}$. Henceforth, we refer to $v(\mathbf{c})$ as the *primal problem*. Furthermore, we call the constraints under the sup-symbol the *conical constraints* and the other constraints the *integral constraints*. A measure μ that satisfies all constraints is called *feasible*. Notice that the set of all feasible measures is typically not a cone. The conical property of $\mathcal{M}$ is mainly imposed to allow a transformation of the general problem to a better tractable problem, which will be introduced below. In fact, the conical assumption can easily be nullified by imposing an integral constraint. A feasible μ that attains the supremum of the problem $v(\mathbf{c})$ is a *solution*. Notice that there exists a collection of problems $v(\mathbf{c})$, one for each $\mathbf{c} \in \mathbb{R}^n$. By problem v we denote this collection of problems. Note the generality of the problem v.

The primal maximization problem can be shown to be associated with a *dual* minimization problem of linear programming type. In many cases, solving the dual problem amounts to determining the convex hull of a set in $\mathbb{R}^n$, specified by the constraints of the primal problem.

[1] Etienne's full name was "De Vijlder, F.E.C. (Florian Etienne Charles)", but for references to his papers we use his *nom de plume* "De Vylder".

[2] There are three named "classical" moment problems: the *Hamburger* moment problem in which the support of the distribution is allowed to be the whole real line, the *Stieltjes* moment problem, for $[0, +\infty)$, and the *Hausdorff* moment problem for a bounded interval, which without loss of generality may be taken as $[0, 1]$.