Available online at www.sciencedirect.com
ScienceDirect
Insurance: Mathematics and Economics 42 (2008) 560–577
INSURANCE MATHEMATICS & ECONOMICS
www.elsevier.com/locate/ime

Integrated insurance risk models with exponential Lévy investment

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Received September 2006; received in revised form June 2007; accepted 29 June 2007

Abstract

We consider an insurance risk model for the cashflow of an insurance company, which invests its reserve into a portfolio consisting of risky and riskless assets. The price of the risky asset is modeled by an exponential Lévy process. We derive the integrated risk process and the corresponding discounted net loss process. We calculate certain quantities as characteristic functions and moments. We also show under weak conditions stationarity of the discounted net loss process and derive the left and right tail behavior of the model. Our results show that the model carries a high risk, which may originate either from large insurance claims or from the risky investment.

MSC: primary 60G51; 62P05; secondary 91B28; 91B30

Keywords: Continuous time perpetuity; Discounted net loss process; Exponential Lévy process; Generalized Ornstein–Uhlenbeck process; Integrated insurance risk process; Integrated risk management; Stochastic recurrence equations; Tail behavior

1. Introduction

Throughout this paper let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a filtered complete probability space on which all stochastic quantities are defined. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and all stochastic processes to be defined in this paper are adapted. We define first the *insurance risk process* as in the Cramér–Lundberg model by

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

where $u > 0$ is the *initial risk reserve*, $c > 0$ is the constant *premium rate* and the *total claim amount process* is defined as compound Poisson process $S(t) = \sum_{j=1}^{N(t)} Y_j$, $t \geq 0$. The claim sizes $(Y_j)_{j\in\mathbb{N}}$ are independent and identically distributed (iid) random variables (r.v.'s) with common distribution function F supported on the whole of $\mathbb{R}^+ = (0, \infty)$ and finite mean μ. The claims arrive at random time points $0 < T_1 < T_2 < \cdots$ and the *claim arrival process* $N(t) = \text{card}\{k \geq 1 : T_k \leq t\}$ for $t > 0$ with $N(0) = 0$ is a homogeneous Poisson process with intensity $\lambda > 0$. Finally, $(N(t))_{t\geq 0}$ and $(Y_j)_{j\in\mathbb{N}}$ are independent processes.

This classical model is extended by allowing for investment of the risk reserve. We consider an insurer who invests its reserve into a Black–Scholes type market consisting of a *bond* and some *stock*, modeled by an exponential Lévy process. Their respective price processes follow the equations

$$X_0(t) = \mathrm{e}^{\delta t} \quad \text{and} \quad X_1(t) = \mathrm{e}^{L(t)}, \quad t \geq 0. \tag{1.1}$$

The constant $\delta > 0$ is the *riskless interest rate*. The process $(L(t))_{t\geq 0}$ is a Lévy process with *characteristic exponent* Ψ, i.e. $E[\exp(\mathrm{i}sL(t))] = \exp(t\Psi(s))$, $s \in \mathbb{R}$, $t \geq 0$, where Ψ has Lévy–Khintchine representation

$$\Psi(s) = \mathrm{i}s\gamma - \frac{\sigma^2}{2}s^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}sx} - 1 - \mathrm{i}sx1_{\{|x|\leq 1\}}\right) \nu(\mathrm{d}x), \quad s \in \mathbb{R}, \tag{1.2}$$

with $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and *Lévy measure* ν satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. The *characteristic triplet* (γ, σ^2, ν) determines the Lévy process. For general Lévy process theory we refer to the monographs by Cont and Tankov (2004) or Sato (1999).

For allocation of the reserve among the riskless and the risky asset we use the so-called *constant mix strategy;* i.e.

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0167-6687/$ - see front matter
doi:10.1016/j.insmatheco.2007.06.002

the initial proportions which are invested into bond and stock remain constant over a predetermined planning horizon; see e.g. Emmer et al. (2001), Section 2. Such a strategy is dynamic in the sense that it requires at every instance of time a rebalancing of the portfolio depending on the corresponding price changes. We denote by $\theta \in [0, 1]$ the fraction of the reserve invested into the risky asset; we call θ the *investment strategy*.

To derive the investment process we follow the calculations in Emmer et al. (2001) and Emmer and Klüppelberg (2004). We state first the corresponding SDEs for the price processes, where we use Itô's formula:

$$\begin{aligned} \mathrm{d}X_0(t) &= \delta X_0(t)\,\mathrm{d}t, \quad t > 0,\ X_0(0) = 1, \\ \mathrm{d}X_1(t) &= X_1(t-)\,\mathrm{d}\widehat{L}(t) \\ &= X_1(t-)\left(\mathrm{d}L(t) + \frac{\sigma^2}{2}\,\mathrm{d}t + \mathrm{e}^{\Delta L(t)} - 1 - \Delta L(t)\right), \\ &\quad t > 0,\ X_1(0) = 1, \end{aligned}$$

where $\Delta L(t, \omega) = L(t, \omega) - L(t-, \omega)$ for each $\omega \in \Omega$ denotes the jump of L at time $t > 0$. The process $\widehat{L}$ is such that $\mathrm{e}^{L(t)} = \mathcal{E}(\widehat{L}(t))$, $t \geq 0$, where $\mathcal{E}$ denotes the stochastic exponential of a process (see, e.g. Protter (1990), Section 2.8, or Cont and Tankov (2004), Section 8.4.2).

Definition 1.1. For $\theta \in [0, 1]$ we define the *investment process* as the solution to the SDE

$$\begin{aligned} \mathrm{d}X_\theta(t) &= X_\theta(t-)\left((1-\theta)\delta\mathrm{d}t + \theta\mathrm{d}\widehat{L}(t)\right), \\ &\quad t > 0,\ X_\theta(0) = 1. \end{aligned} \tag{1.3}$$

This approach is based on self-financing portfolios and hence classical in financial portfolio optimization; see Korn (1997), Section 2.1. The following is a consequence of Itô's Lemma.

Lemma 1.2. *The SDE (1.3) has the solution*

$$X_\theta(t) = \mathcal{E}(\widehat{L}_\theta(t)) = \mathrm{e}^{L_\theta(t)}, \quad t \geq 0, \tag{1.4}$$

where $\widehat{L}_\theta(t) = (1-\theta)\delta t + \theta\widehat{L}(t)$ *and* L_θ *is such that* $\mathcal{E}(\widehat{L}_\theta(t)) = \mathrm{e}^{L_\theta(t)}$.

Lemma 1.3 (*Emmer and Klüppelberg (2004), Lemma 2.5*). *The process* $(L_\theta(t))_{t\geq 0}$ *is a Lévy process with characteristic exponent* Ψ_θ, *and the characteristic triplet* $(\gamma_\theta, \sigma_\theta^2, \nu_\theta)$ *is given by*

$$\begin{aligned} \gamma_\theta &= \gamma\theta + (1-\theta)\left(\delta + \frac{\sigma^2}{2}\theta\right) \\ &\quad + \int_{\mathbb{R}} (\log(1 + \theta(\mathrm{e}^x - 1))1_{\{|\log(1+\theta(\mathrm{e}^x-1))|\leq 1\}} \\ &\quad - \theta x 1_{\{|x|\leq 1\}})\nu(\mathrm{d}x), \\ \sigma_\theta^2 &= \theta^2\sigma^2, \\ \nu_\theta(A) &= \nu\left(\left\{x \in \mathbb{R} : \log(1 + \theta(\mathrm{e}^x - 1)) \in A\right\}\right) \\ &\quad \textit{for any Borel set } A \subset \mathbb{R}. \qquad \square \end{aligned}$$

Remark 1.4. (i) Besides the characteristic exponents Ψ and Ψ_θ we shall also need the *Laplace exponents* given by

$$\varphi(s) = \Psi(\mathrm{i}s) = \log E\left[\mathrm{e}^{-sL(1)}\right], \tag{1.5}$$

$$\varphi_\theta(s) = \Psi_\theta(\mathrm{i}s) = \log E\left[\mathrm{e}^{-sL_\theta(1)}\right], \tag{1.6}$$

provided they exist. If $\varphi(s) < \infty$, then $E\left[\mathrm{e}^{-sL(t)}\right] = \mathrm{e}^{t\varphi(s)} < \infty$ for all $t \geq 0$, see Sato (1999), Theorem 25.17. As we show in Lemma A.1(c), $E\left[\mathrm{e}^{sL_\theta(1)}\right] < \infty$ for all $\theta \in [0, 1]$ provided $E\left[\mathrm{e}^{sL(1)}\right] < \infty$.
(ii) A jump of size ΔL of L leads to a jump of size $\mathrm{e}^{\Delta L} - 1$ of $\widehat{L}$ and to a jump of size $\Delta L_\theta = \log(1 + \theta(\mathrm{e}^{\Delta L} - 1)) > \log(1-\theta)$ of L_θ. In other words, ν_θ is the image measure of ν under the transformation $x \mapsto \log(1 + \theta(\mathrm{e}^x - 1))$. This explains the requirement $\theta \leq 1$.
(iii) If L is a process of finite variation, then L_θ is as well. Indeed,

$$\begin{aligned} &\int_{|x|\leq 1} |x|\nu_\theta(\mathrm{d}x) \\ &\quad = \int_{|\log(1+\theta(\mathrm{e}^x-1))|\leq 1} |\log(1 + \theta(\mathrm{e}^x - 1))|\nu(\mathrm{d}x) \\ &\quad \leq \int_{-\infty}^{-1} |\log(1 + \theta(\mathrm{e}^x - 1))|\nu(\mathrm{d}x) \\ &\qquad + \int_{-1}^{p} |\log(1 + \theta(\mathrm{e}^x - 1))|\nu(\mathrm{d}x), \end{aligned}$$

where $p = \log(1 + \theta^{-1}(e - 1)) > 0$. Then $\int_{-\infty}^{-1} |\log(1 + \theta(\mathrm{e}^x - 1))|\nu(\mathrm{d}x) \leq |\log(1-\theta)| \int_{-\infty}^{-1} \nu(\mathrm{d}x) < \infty$ and, because of the finite variation of L, also $\int_{-1}^{p} |\log(1+\theta(\mathrm{e}^x-1))|\nu(\mathrm{d}x) \leq \int_{-1}^{p} |x|\nu(\mathrm{d}x) < \infty$ holds. $\square$

The goal of this paper is to study the integrated risk process, which allows for risk assessment of the insurance and investment risk at the same time. This process is defined in Section 2. We assume throughout this paper that investment process and total claim amount process are independent, which allows for a very explicit analysis of the integrated risk process.

In Section 3 the stationary version of the integrated risk process, the discounted net loss process (DNLP), is defined and investigated. The model fits into the framework of *generalized Ornstein–Uhlenbeck processes*, which have recently attracted much attention. Due to the special structure of our model we derive more specific results than in the more general case treated in Lindner and Maller (2005). We start with stationarity conditions and compare the process to its natural embedded discrete skeleton process; i.e. the process sampled at the claim arrival times. Our most important results in this section concern the tail behavior of the stationary distribution. We show in particular that the stationary distribution of the continuous time process and the discrete time process coincide. We analyse two different regimes, which both lead to Pareto tails of the stationary distribution. The reasons, however, are different. If the claims have finite moments of sufficiently high order, under weak regularity conditions, both tails of the stationary DNLP