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A generalization of the credibility theory obtained by using the weighted balanced loss function

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Abstract

In this paper an alternative to the usual credibility premium that arises for weighted balanced loss function is considered. This is a generalized loss function which includes as a particular case the weighted quadratic loss function traditionally used in actuarial science. From this function credibility premiums under appropriate likelihood and priors can be derived. By using weighted balanced loss function we obtain, first, generalized credibility premiums that contain as particular cases other credibility premiums in the literature and second, a generalization of the well-known distribution free approach in [Bühlmann, H., 1967. Experience rating and credibility. Astin Bull. 4 (3), 199–207].

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1. Introduction

Credibility theory is a set of quantitative methods which allows an insurer to adjust future premiums based on past experience. Generally, the credibility expression obtained is written as a weighted sum of the sample mean and the collective premium, the premium to be charged to a group of policyholders in a portfolio. The weighted factor is referred as the credibility factor. Some historical references on this topic are Whitney (1918), Mowbray (1914), Bailey (1945), Bühlmann (1967), Kahn (1975), Gerber and Arbor (1980), Eichenauer et al. (1988), Heilmann (1989), Goovaerts et al. (1990) and Herzog (1996). For a recent revision of the credibility theory see Landsman and Makov (1999, 2000), Promislow and Young (2000), Young (2000) and Gómez et al. (2006).

It is well-known that completely different methods can lead to the same expression of the credibility factor. These methods are, among others, the distribution free approach (Bühlmann, 1967), Bayesian methods (Bailey (1945), Heilmann (1989) and Herzog (1996); among others) and under the Bayesian methodology the Γ–minimax (Eichenauer et al., 1988) and the posterior regret Γ–minimax approaches (Gómez et al., 2006).

In Heilmann (1989) many credibility premiums were obtained under statistical decision theory from a Bayesian point of view and using appropriate weighted squared-error loss function (WLF henceforth), $L_1(a, x) = h(x)(x - a)^2$, and pairs of likelihood and prior distributions (usually known in actuarial practice as structure function). By using different functional forms for $h(x)$ we have different premium principles. For example for $h(x) = 1$ and $h(x) = \exp\{cx\}$, $c > 0$, we have the net and Esscher premium principles (Heilmann (1989) and Gómez et al. (2006); among others), respectively.

It is well-known (see Jewell (1974)) that for the exponential family of distributions and its conjugate priors exact net credibility premiums are obtained and also for the par Poisson–gamma under the Esscher premium (see Heilmann (1989)). In this case, the Bayes premium can be written as a credibility formula in the form:

$$P_B^{L_1} = Z(t)g(\bar{\mathbf{x}}) + [1 - Z(t)]P_C^{L_1}, \tag{1}$$

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where we have denoted by $P_B^{L_1}$ and $P_C^{L_1}$ the Bayes and collective premium obtained under WLF (see Heilmann (1989) and Gómez et al. (2006) for details) and $g(\bar{\mathbf{x}})$ is a function of the observed data.

In this paper, generalizations of the credibility premiums are derived using the general weighted balanced loss function (WBLF henceforth) introduced here as

$$L_2(a, x) = wh(x)(\delta_0(x) - a)^2 + (1 - w)h(x)(x - a)^2, \quad (2)$$

where $0 \leq w \leq 1$ is a weighting factor determined by the practitioner, $h(x)$ is a positive weight function and $\delta_0(x)$ is a function of the observed data.

WBLF is a generalized loss function introduced in Zellner (1994, pp. 371–390), and which appears also in Dey et al. (1999), Farsipour and Asgharzadhe (2004) and Jafari et al. (2006) by taking $h(x) = 1$ in (2). This loss includes as a particular case of the WLF when w is chosen as equal to 0.

Furthermore, using this WBLF a generalization of the credibility expression in Bühlmann (1967) under the distribution free approach is also obtained.

Section 2 includes the methodology used to derive premiums under WBLF, which is similar to the one in Heilmann (1989). In Section 3 some credibility premiums obtained as particular cases are derived and a short extension of the Esscher principle is shown. Solution under the distribution free approach is presented in Section 4. Finally, conclusions are presented in Section 5.

2. The methodology

In this section, new credibility formula under the net premium principle are derived by using the WBLF in (2). For that reason, we assume that the individual risk, X, has a density $f(x|\theta)$, indexed by a parameter $\theta \in \Theta$ which has a prior distribution with density $\pi(\theta)$. Let, now, $\pi^{\mathbf{x}}(\theta)$ be the posterior density when $\mathbf{x}$ is observed.

In actuarial literature, the unknown risk premium $P_R^L \equiv P_R^L(\theta)$ is obtained by minimizing the expected loss $\mathbb{E}_f[L(\theta, P)]$ for some loss function L. If experience is not available, the actuary charges the collective premium P_C^L, which is given by minimizing the risk function, i.e. minimizing $\mathbb{E}_\pi[L(P_R^L(\theta), P_C^L)]$. Finally, if experience is available, the actuary computes the Bayes premium which is computed in the same way as the collective premium by interchanging the prior by the posterior distribution. Next proposition is a generalization of Lemma 2 in Jafari et al. (2006) from which the Bayes estimator of θ under prior π is obtained.

Proposition 1. *Under WBLF and prior π, the risk and collective premiums are given by*

$$P_R^{L_2}(\theta) \equiv P_R^{L_2} = w\frac{\mathbb{E}_{f(x|\theta)}[\delta_0(X)h(X)|\theta]}{\mathbb{E}_{f(x|\theta)}[h(X)|\theta]} + (1-w)\frac{\mathbb{E}_{f(x|\theta)}[Xh(X)|\theta]}{\mathbb{E}_{f(x|\theta)}[h(X)|\theta]}, \quad (3)$$

$$P_C^{L_2} = w\delta_0^* + (1-w)\frac{\mathbb{E}_\pi[P_R^{L_2}h(P_R^{L_2})]}{\mathbb{E}_\pi[h(P_R^{L_2})]}, \quad (4)$$

respectively and where δ_0^* is a target estimator for the risk premium $P_R^{L_2}$.

Proof. The proof is straightforward minimizing $\mathbb{E}_{f(x|\theta)}[L_2(\theta, P_R^{L_2})]$ and $\mathbb{E}_{\pi(\theta)}[L_2(P_R^{L_2}, P_C^{L_2})]$ with respect to $P_R^{L_2}$ and $P_C^{L_2}$ to obtain the risk and collective premium, respectively. □

Now the Bayes premium, $P_B^{L_2}$, is obtained replacing in (4) $\pi(\theta)$ by $\pi^{\mathbf{x}}(\theta)$.

Observe that by putting $\gamma = \mathbb{E}_\pi[P_R^{L_2}h(P_R^{L_2})]/\mathbb{E}_\pi[h(P_R^{L_2})]$,

$$P_C^{L_2} \in (\delta_0^*, \gamma), \quad \text{if } \delta_0^* < \gamma,$$
$$P_C^{L_2} \in (\gamma, \delta_0^*), \quad \text{if } \delta_0^* > \gamma,$$

and the same result occurs when C is replaced by B. Therefore, the actuary can choose the value of δ_0^* to obtain a premium according to his preferences.

Next proposition provides the net credibility premium under WBLF, i.e. we are assuming that $h(x) = 1$.

Proposition 2. *If the Bayes net premium obtained under $L_1(a, x)$ is a credibility formula, the Bayes balanced net premium obtained under WBLF is also a credibility formula in the form:*

$$P_B^{L_2} = Z(t)l(P_C^{L_1}) + [1 - Z(t)]l(\bar{\mathbf{x}}),$$

where $Z(t) \in [0, 1]$ and $l(x) = (1-w)^2x + w(1-w)\mathbb{E}_{\pi^{\mathbf{x}}(\theta)}[E_{f(x|\theta)}(\delta_0(X|\theta))] + w\delta_0^*$.

Proof. Using (3) and (4) with $h(x) = 1$ we have that

$$P_R^{L_2} = w\mathbb{E}[\delta_0(X)|\theta] + (1-w)\mathbb{E}_{f(x|\theta)}(X|\theta)$$

and

$$\begin{aligned} P_C^{L_2} &= w\delta_0^* + (1-w)\mathbb{E}_{\pi(\theta)}[w\mathbb{E}_{f(x|\theta)}[\delta_0(X)|\theta] \\ &\quad + (1-w)\mathbb{E}_{f(x|\theta)}(X|\theta)] \\ &= w\delta_0^* + w(1-w)\mathbb{E}_{\pi(\theta)}\{\mathbb{E}_{f(x|\theta)}[\delta_0(X|\theta)]\} \\ &\quad + (1-w)^2\mathbb{E}_{\pi(\theta)}[\mathbb{E}_{f(x|\theta)}\delta_0(X)|\theta]. \end{aligned}$$

Therefore

$$\begin{aligned} P_B^{L_2} &= w\delta_0^* + w(1-w)\mathbb{E}_{\pi^{\mathbf{x}}(\theta)}\{\mathbb{E}_{f(x|\theta)}[\delta_0(X)|\theta] \\ &\quad + (1-w)^2P_B^{L_1}\}. \end{aligned}$$

Now, if $P_B^{L_1}$ is a credibility formula in the form

$$P_B^{L_1} = Z(t)P_C^{L_1} + [1 - Z(t)]\bar{\mathbf{x}},$$

then

$$\begin{aligned} P_B^{L_2} &= w\delta_0^* + w(1-w)\mathbb{E}_{\pi^{\mathbf{x}}(\theta)}\{\mathbb{E}_{f(x|\theta)}[\delta_0(X)|\theta]\} \\ &\quad + (1-w)^2\{Z(t)P_C^{L_1} + [1 - Z(t)]\bar{\mathbf{x}}\} \\ &= Z(t)[(1-w)^2P_C^{L_1} + w(1-w)\mathbb{E}_{\pi^{\mathbf{x}}(\theta)} \\ &\quad \times \{\mathbb{E}_{f(x|\theta)}[\delta_0(X)|\theta]\} + w\delta_0^*] \\ &\quad + [1 - Z(t)][(1-w)^2\bar{\mathbf{x}} + w\delta_0^* \\ &\quad + w(1-w)\mathbb{E}_{\pi^{\mathbf{x}}(\theta)}\{\mathbb{E}_{f(x|\theta)}[\delta_0(X)|\theta]\}] \\ &= Z(t)l(P_C^{L_1}) + [1 - Z(t)]l(\bar{\mathbf{x}}). \quad \square \end{aligned}$$