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A hidden Markov regime-switching model for option valuation

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ABSTRACT

We investigate two approaches, namely, the Esscher transform and the extended Girsanov's principle, for option valuation in a discrete-time hidden Markov regime-switching Gaussian model. The model's parameters including the interest rate, the appreciation rate and the volatility of a risky asset are governed by a discrete-time, finite-state, hidden Markov chain whose states represent the hidden states of an economy. We give a recursive filter for the hidden Markov chain and estimates of model parameters using a filter-based EM algorithm. We also derive predictors for the hidden Markov chain and some related quantities. These quantities are used to estimate the price of a standard European call option. Numerical examples based on real financial data are provided to illustrate the implementation of the proposed method.

1. Introduction

Regime-switching models are an important class of financial time series models. A key feature of regime-switching models is that model parameters are functions of a hidden Markov chain whose states represent hidden states of an economy, or different stages of business cycles. Consequently, regime-switching models can incorporate structural changes of economic conditions. The history of the regime-switching models can be traced back to the early works of Quandt (1958) and Goldfeld and Quandt (1973), where a class of regime-switching regression models was applied to model nonlinear economic data. The idea of regime-switching also appeared in some early works of nonlinear time series analysis, (see Tong (1983)). Hamilton (1989) pioneered applications of regime-switching models in economics and econometrics. Empirical studies reveal that regime-switching models fit economic and financial time series well and explain some important stylized facts of these series. Moreover, regime-switching models have diverse applications in finance. Some of these applications include Pliska (1997) and Elliott et al. (2001) for short rate models, Elliott and Hinz (2002) for portfolio analysis, Naik (1993), Guo (2001) and Elliott et al. (2005) for option valuation, Elliott et al. (1998) for volatility estimation, and others.

Recently there is a growing interest in the use of regime-switching models for option valuation. Regime-switching models incorporate the impact of structural changes in economic conditions on option valuation. This is particularly important for valuing long-lived options, such as options embedded in equity-linked securities and participating life insurance products. However, because of the additional source of uncertainty induced by regime-switching, the market in a regime-switching model is, in general, incomplete. Consequently, there is more than one equivalent martingale measure for valuation. In this case, the standard Black-Scholes-Merton option pricing argument cannot be applied and the question of which equivalent martingale measure one should choose for valuation becomes important. Different methods have been developed to value options in an incomplete market. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996) introduced the minimization of a quadratic function of hedging errors for valuation. Hodges and Neuberger (1989) developed a utility-based indifference pricing approach in an incomplete market. Davis (1997) used traditional economic equilibrium arguments to value options and formulated the problem as a utility maximization problem. Gerber and Shiu (1994)

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pioneered the use of the Esscher transform, a well-known tool in actuarial science, to value options in an incomplete market. The Esscher transform provides a convenient way to select an equivalent martingale measure. Gerber and Shiu justified the use of the Esscher transform for option valuation by the maximization of an expected power utility of an economic agent. Elliott and Madan (1998) introduced an extended Girsanov's principle to select an equivalent martingale measure in a discrete-time financial model. The extended Girsanov's principle provides a general method to value options under discrete-time econometric time series models. Badescu et al. (2009) established a relationship between the Esscher transform valuation principle, the extended Girsanov's valuation principle and consumption-based equilibrium asset pricing models.

Some methods for option valuation specifically geared to regime-switching models have been introduced in the literature. Guo (2001) used a set of "fictitious" assets, namely, change-of-state contracts, to complete a continuous-time, regime-switching market. The theoretical basis of these change-of-states contracts is the Arrow-Debreu securities. Elliott et al. (2005) proposed the use of the Esscher transform to value options in a continuous-time, regime-switching economy and justified its use by the minimal martingale entropy measure. Siu (2008) further justified the Esscher transform approach for option valuation in a continuous-time regime-switching model using a saddle-point result, (a special case of the Nash equilibrium), arising from a two-person, zero-sum, stochastic differential game. Most previous work assumes that the Markov chain modulating a regime-switching model is observable. However, in practice, the "true" state of an underlying economy may not be observed. Therefore, it is of practical relevance to relax the assumption that the chain is observable. Ishijima and Kihara (2005) studied the option valuation problem in a discrete-time regime-switching model governed by a hidden Markov chain. They employed the locally risk-neutral valuation relationship of Duan (1995) to determine an equivalent martingale measure for valuation.

In this paper, we investigate an option valuation problem in a discrete-time hidden Markov regime-switching Gaussian model. The model's parameters, including the market interest rate, the appreciation rate and the volatility of a risky asset are governed by a discrete-time, hidden Markov chain. The states of the chain represent different states of an economy. We consider below two approaches to determine an equivalent martingale measure. First, we consider the use of the Esscher transform to choose an equivalent martingale measure. This choice is justified by the maximization of an expected power utility of an economic agent. Second we study an extended Girsanov's principle for selecting an equivalent martingale measure. It is shown that the two approaches lead to the same pricing result. We give a recursive filter for the hidden Markov chain and estimates of model parameters using a filter-based EM algorithm. We also derive predictors for the hidden Markov chain and some related quantities. These quantities are used to estimate a price of a standard European call option. Numerical examples based on real financial data are given to illustrate the implementation of the proposed method. We also provide numerical comparisons of the European call prices obtained from the proposed estimation method, the call prices arising from an analytic formula and from the Black-Scholes-Merton model.

The approach considered here is different from that in Ishijima and Kihara (2005). We adopt the Esscher transform while the valuation method in Ishijima and Kihara (2005) is based on the local risk-neutral valuation considered Duan (1995), which may be traced back to an economic equilibrium approach for asset pricing in a pure exchange economy pioneered by Lucas (1978). Indeed, the Esscher transform provides a more flexible way to price options than the local risk-neutral valuation approach; the former can be applied to any return distribution with a finite moment generation function while the latter can be used only when the return distribution is normal. However, for illustration we consider only the Esscher transform approach for option valuation in a hidden Markov regime-switching Gaussian model. Although the same valuation principle can be applied to a general hidden Markov regime-switching non-Gaussian model. This may provide a possible topic for future research. We also justify the use of the Esscher transform to option valuation using the extended Girsanov's principle in Elliott and Madan (1998), which is supported by weak-form efficient hedging strategies minimizing the variance of risk-adjusted costs of hedging. We also establish the consistency between the Esscher transform approach, the local-risk-neutral-valuation approach, the extended Girsanov principle and the utility maximization approach in the context of hidden Markov asset price models. This consistency was not explored in Ishijima and Kihara (2005). Furthermore, we adopt a different filtering approach to estimate the hidden states and the parameters of the hidden Markov model. The filtering methods considered here are based on those developed in Elliott et al. (1994). Finally, we derive an estimate for a price for a standard European call option which is more easy to implement than that in Ishijima and Kihara (2005).

The paper is organized as follows: The next section presents the discrete-time, hidden Markov regime-switching Gaussian model. In Section 3, we discuss the use of the Esscher transform and the extended Girsanov's principle to determine equivalent martingale measures. In Section 4, we derive filters and predictors that are required to derive an estimate for the price of an option. Section 5 gives the estimate for the price based on observed price information. Section 6 presents and discusses the numerical examples. The final section summarizes the results.

2. The model

In this section, we present the hidden Markov regime-switching Gaussian model for asset prices in a discrete-time economy. Let $\mathcal{T}$ be the time index set $\{0, 1, 2, \ldots, T\}$, where $T < \infty$, which represents time points at which economic activities take place. In our simplified world, the economy has two primitive securities, namely, a bond and a risky asset. These securities can be traded over time in the horizon $\mathcal{T}$. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability measure. We suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to incorporate uncertainties due to fluctuations in market prices and changes in economic conditions over time.

First, we describe the evolution of the hidden state of the economy over time. Let $\mathbf{X} := \{\mathbf{X}_t \mid t \in \mathcal{T}\}$ be a discrete-time, finite-state, hidden Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N\}$. Without loss of generality, as in Elliott et al. (1994), we identify the state space of the Markov chain $\mathbf{X}$ with the finite set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)' \in \Re^N$ so $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$, the Kronecker delta. Here $\mathbf{y}'$ is the transpose of a vector, or a matrix, $\mathbf{y}$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\Re^N$. We call $\mathcal{E}$ the canonical state space of the chain $\mathbf{X}$.

We suppose further that the Markov chain $\mathbf{X}$ is time-homogeneous. The probability law of $\mathbf{X}$ is specified by its transition probabilities and initial distribution. For each $i, j = 1, 2, \ldots, N$, let

$$a_{ji} := \mathbb{P}(\mathbf{X}_{t+1} = \mathbf{e}_j \mid \mathbf{X}_t = \mathbf{e}_i).$$

Write $\mathbf{A}$ for the transition probability matrix $[a_{ji}]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{X}$ under $\mathbb{P}$. Let $\boldsymbol{\pi} := (\pi_1, \pi_2, \ldots, \pi_N)' \in \Re^N$, where

$$\pi_i := \mathbb{P}(\mathbf{X}_0 = \mathbf{e}_i),$$

so that $\boldsymbol{\pi}$ is the initial distribution of the chain $\mathbf{X}$. We suppose that the chain $\mathbf{X}$ is stationary.