Insurance: Mathematics and Economics 47 (2010) 428–433

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

On the DFR property of the compound geometric distribution with applications in risk theory

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ARTICLE INFO

Article history:
Received March 2010
Received in revised form September 2010
Accepted 15 September 2010

MSC:
primary 60K05
secondary 91B30

Keywords:
Compound geometric distribution
Renewal equation
DFR (IFR)
NWU (NBU)
IMRL (DMRL)
NWUE (NBUE)
Ladder height
Ruin probability

ABSTRACT

In 1988, Shanthikumar proved that the sum of a geometrically distributed number of i.i.d. DFR random variables is also DFR. In this paper, motivated by the inverse problem, we study monotonicity properties related to defective renewal equations, and obtain that if a compound geometric distribution is DFR, then the random variables of the sums are NWU (a class that contains DFR). Furthermore, we investigate some applications of risk theory and give a characterization of the exponential distribution.

1. Introduction and preliminaries

Let $X_1, X_2, \ldots$ be an independent and identically distributed (i.i.d.) sequence of random variables supported on $[0, \infty)$, with continuous distribution function (d.f.) $F(x) = \Pr(X \le x)$, $F(0) = 0$ and mean $\mu < \infty$. Let also N be a counting random variable independent of X_i, $i = 1, 2, \ldots$, with $\Pr(N = n) = (1 - \phi)\phi^n$, $n = 0, 1, 2, \ldots$ $(0 < \phi < 1)$. Thus, the compound geometric distribution $G(x)$ of the random variable $S_N = X_1 + X_2 + \cdots + X_N$ is

$$G(x) = \Pr(S_N \le x) = (1-\phi)\sum_{n=0}^{\infty}\phi^n F^{*n}(x), \quad x \ge 0, \tag{1}$$

where $F^{*n}(x) = \Pr(X_1 + X_2 + \cdots + X_n \le x)$ is the nth-fold convolution of F with itself. The density of G is

$$g(x) = (1-\phi)\sum_{n=1}^{\infty}\phi^n f^{*n}(x), \quad x \ge 0,$$

where $f(x)$ is the density associated with $F(x)$.

Compound geometric distributions play an important role in terminating renewal processes, reliability, queueing, branching processes and insurance (see Feller (1971), Gertsbakh (1984), Kalashnikov (1997), Rolski et al. (1999), Asmussen (2000), Willmot and Lin (2001) and references therein). For example, the equilibrium waiting time in the $G/G/1$ queue and the probability of non-ruin in the renewal risk model have a compound geometric distribution.

The tail of $G(x)$ is defined by

$$\overline{G}(x) = 1 - G(x) = (1-\phi)\sum_{n=1}^{\infty}\phi^n \overline{F^{*n}}(x), \tag{2}$$

where $\overline{F^{*n}}(x) = 1 - F^{*n}(x)$. This function satisfies the defective renewal equation (d.r.e.)

$$\overline{G}(x) = \phi\int_0^x \overline{G}(x-t)\mathrm{d}F(t) + \phi\overline{F}(x), \tag{3}$$

whose solution is

$$\overline{G}(x) = \frac{\phi}{1-\phi}\int_{0^-}^x \overline{F}(x-t)\mathrm{d}G(t), \tag{4}$$

where the interval in the range of integration above is closed (see Willmot and Lin (2001, p. 156)). Note that $G(x)$ has a mass point at 0, $G(0) = 1 - \phi$.

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doi:10.1016/j.insmatheco.2010.09.005

Next, we recall the definition of some reliability classes of distributions. A d.f. F with tail $\overline{F}(x) = 1 - F(x)$, $x \geq 0$, is said to be *decreasing (increasing) failure rate* or DFR (IFR) if $\overline{F}(x+y)/\overline{F}(x)$ is nondecreasing (nonincreasing) in x for any $y \geq 0$. If F is absolutely continuous with density f, then it is DFR (IFR) when the failure rate $\lambda_F(x) = f(x)/\overline{F}(x)$ is nonincreasing (nondecreasing). The exponential is the only distribution that is both DFR and IFR, and a mixture of exponential d.f. is always DFR, see Willmot and Lin (2001, Chapter 2).

A d.f. F is called *new worse (better) than used* or NWU (NBU) if $\overline{F}(x+y) \geq (\leq) \overline{F}(x)\overline{F}(y)$ for every $x, y \geq 0$. The DFR (IFR) class of distributions is a subclass of the NWU (NBU) class. The mean residual lifetime of F is defined by $r_F(x) = \int_x^\infty \overline{F}(y)\mathrm{d}y/\overline{F}(x)$. A d.f. F is called *increasing (decreasing) mean residual lifetime* or IMRL (DMRL) if $r_F(x)$ is nondecreasing (nonincreasing) in x. The DFR (IFR) class is also a subclass of the IMRL (DMRL) class. Finally, F is *new worse (better) than used in expectation* or NWUE (NBUE) if $\int_x^\infty \overline{F}(t)\mathrm{d}t \geq (\leq) \mu \overline{F}(x)$ for every $x \geq 0$. The IMRL (DMRL) and NWU (NBU) classes are both subclasses of the NWUE (NBUE) class. The suggested references on these classes are the books by Barlow and Proschan (1981) and Willmot and Lin (2001).

In the paper of Shanthikumar (1988), it is obtained that *if F is DFR, then G is also DFR*. One of the main purposes of this paper is the study of the inverse problem, namely, the characterization of F under the hypothesis that G is DFR. A similar problem was studied by Chen (1994), who has derived the relation between renewal function and excess lifetime in renewal theory. The paper is organized as follows: In Section 2, we present general inequalities with the use of defective renewal equation, under a hypothesis of ratio monotonicity. Under this perspective, we present in Section 3 some applications in risk theory. In particular, we prove that if G is DFR (IFR), then F is NWU (NBU). We also obtain that if G is IMRL (DMRL), then F is NWUE (NBUE). Moreover, we give a new characterization of the exponential distribution, using the compound geometric distribution. Finally, in Section 4 we discuss through an example that G can be IFR.

2. Main results

Let $m(x)$ be a continuous nonnegative function that satisfies the d.r.e.

$$m(x) = \phi \int_0^x m(x-t)\mathrm{d}F(t) + \upsilon(x), \tag{5}$$

where $0 < \phi < 1$ and $\upsilon(x) \geq 0$ is locally bounded. The general form for the solution of $m(x)$ in (5), which is vanishing for $x < 0$ and bounded on finite intervals, is given by

$$m(x) = \frac{1}{1-\phi}\int_{0-}^x \upsilon(x-t)\mathrm{d}G(t), \tag{6}$$

where the interval in the range of integration above is closed (see Asmussen (1987, Chapter VI)). We also consider a function $m(x, y)$ that satisfies the d.r.e.

$$m(x,y) = \phi \int_0^x m(x-t,y)\mathrm{d}F(t) + \upsilon(x+y), \tag{7}$$

whose solution is

$$m(x,y) = \frac{1}{1-\phi}\int_{0-}^x \upsilon(x+y-t)\mathrm{d}G(t). \tag{8}$$

We first give a monotonicity result for the ratio $m(x, y)/m(x+y)$.

Lemma 2.1. *If $g(x+t_1)/m(x+t_2)$ is nonincreasing (nondecreasing) in x for any $t_2 \geq t_1 \geq 0$, then $m(x, y)/m(x+y)$ is nondecreasing (nonincreasing) in x.*

Proof. Let $g(x+t_1)/m(x+t_2)$ be nonincreasing in x for any $t_2 \geq t_1 \geq 0$. By (6) and (8), we have

$$m(x,y) = m(x+y) - \frac{1}{1-\phi}\int_x^{x+y} \upsilon(x+y-t)\mathrm{d}G(t). \tag{9}$$

Dividing by $m(x+y)$, it follows

$$\begin{aligned}\frac{m(x,y)}{m(x+y)} &= 1 - \frac{1}{1-\phi}\int_x^{x+y} \frac{\upsilon(x+y-t)}{m(x+y)} g(t)\mathrm{d}t \\ &= 1 - \frac{1}{1-\phi}\int_0^y \upsilon(y-t)\frac{g(x+t)}{m(x+y)}\mathrm{d}t.\end{aligned} \tag{10}$$

Thus, by hypothesis, the result follows by (10). In the case where $g(x+t_1)/m(x+t_2)$ is nondecreasing in x for any $t_2 \geq t_1 \geq 0$, the proof is similar. □

Using the result of the above lemma and the d.r.e. for $m(x, y)$, we derive inequalities for the function $\upsilon(x)$.

Theorem 2.1. *If $g(x+t_1)/m(x+t_2)$ is nonincreasing (nondecreasing) in x for any $t_2 \geq t_1 \geq 0$ and $m(x+y)/m(x)$ is nondecreasing (nonincreasing) in x for any $y \geq 0$, then*

$$\upsilon(x+y) \geq (\leq) \frac{\upsilon(x)\upsilon(y)}{\upsilon(0)}. \tag{11}$$

Proof. Let $g(x+t_1)/m(x+t_2)$ be nonincreasing in x for any $t_2 \geq t_1 \geq 0$ and $m(x+y)/m(x)$ nondecreasing in x for any $y \geq 0$. Dividing (7) by $m(x+y)$, we obtain

$$\frac{m(x,y)}{m(x+y)} = \phi\int_0^x \frac{m(x-t,y)}{m(x+y)}\mathrm{d}F(t) + \frac{\upsilon(x+y)}{m(x+y)},$$

or equivalently,

$$\frac{m(x,y)}{m(x+y)} = \phi\int_0^x \frac{m(x-t,y)}{m(x+y-t)}\frac{m(x+y-t)}{m(x+y)}\mathrm{d}F(t) + \frac{\upsilon(x+y)}{m(x+y)}.$$

By hypothesis and Lemma 2.1, the ratio $m(x, y)/m(x+y)$ is nondecreasing in x for any $y \geq 0$, and the ratio $m(x+y-t)/m(x+y)$ is nonincreasing in $x \geq t \geq 0$ for any $y \geq 0$. Hence, we have

$$\begin{aligned}\frac{m(x,y)}{m(x+y)} &\leq \phi\frac{m(x,y)}{m(x+y)}\int_0^x \frac{m(x+y-t)}{m(x+y)}\mathrm{d}F(t) + \frac{\upsilon(x+y)}{m(x+y)} \\ &\leq \phi\frac{m(x,y)}{m(x+y)}\int_0^x \frac{m(x-t)}{m(x)}\mathrm{d}F(t) + \frac{\upsilon(x+y)}{m(x+y)}.\end{aligned}$$

By Eq. (5), it follows that

$$\phi\int_0^x \frac{m(x-t)}{m(x)}\mathrm{d}F(t) = 1 - \frac{\upsilon(x)}{m(x)}. \tag{12}$$

Thus,

$$\frac{m(x,y)}{m(x+y)} \leq \frac{m(x,y)}{m(x+y)}\left[1 - \frac{\upsilon(x)}{m(x)}\right] + \frac{\upsilon(x+y)}{m(x+y)},$$

and after some rearrangements, we obtain

$$\frac{\upsilon(x)}{m(x)}\frac{m(x,y)}{m(x+y)} \leq \frac{\upsilon(x+y)}{m(x+y)}.$$

Using again the monotonicity of $m(x, y)/m(x+y)$, it follows

$$\frac{\upsilon(x)}{m(x)}\frac{m(0,y)}{m(0+y)} \leq \frac{\upsilon(x+y)}{m(x+y)},$$

or equivalently, since $m(0, y) = \upsilon(y)$ by (7),

$$\frac{m(x+y)}{m(x)m(y)} \leq \frac{\upsilon(x+y)}{\upsilon(x)\upsilon(y)}. \tag{13}$$