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Modelling total tail dependence along diagonals

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Abstract

An approach to modelling total tail dependence beyond the main diagonals is proposed. The concept introduced combines the principal and minor diagonals to describe total extreme dependence. A framework is introduced for the measurement of total tail dependence under model mixture. Illustrations are presented using empirical data on stock market indices and exchange rates. An extension is provided to the multivariate case and total tail dependence is considered for model mixtures.

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Keywords: Copula; Dependence structure; Model mixture; Quantitative risk; Total tail dependence

1. Introduction

Tail dependence is an important concept in multivariate statistics (Joe, 1997; Schweizer, 1991; Jogdeo, 1982). The theory of copulas can be used to describe tail dependence for underlying assets in the context of quantitative risk management (McNeil et al., 2005; Alink et al., 2004; Embrechts et al., 2003). It is well known that the normal (or Gaussian) copula leads to asymptotic independence provided $\rho < 1$. This exemplifies an insufficiency of Pearson's linear correlation for the description of extreme events (Embrechts et al., 1999; Sibuya, 1960). Mikusiński et al. (1991) applied the concept of the convex sum of copulas and gave a probabilistic interpretation for constructing a new copula. Coles et al. (1999) proposed measures and diagnostics for extremal dependence behavior and discussed the issue of statistical estimation. Heffernan (2000) provided a summary of coefficients for the upper and the lower joint tail dependence and assessed the range of asymptotic dependence for bivariate distributions. Charpentier (2003) introduced the conditional copula for the upper–upper tail dependence and studied the shape of the dependence structure in the tails.

Modelling tail dependence of bivariate random vectors is essential to risk management in general, and to market regulation in particular (Frahm et al., 2005; Longin and Solnik, 2001; Embrechts et al., 1999). The association of underlying market assets (e.g. stocks, futures and commodities) must be measured as a function of volatility (Dropsy and Fatemeh, 1994), especially when a financial crisis arises in the case of financial contagion in emerging markets (Rodriguez, 2006; Longin and Solnik, 2001). Empirical studies have shown that typically, log returns of financial assets are non-Gaussian. For example, mixture models were considered by Zhang and Cheng (2004), Lo and MacKinlay (1999), McLanchlan and Krishnan (1997). In this paper, we apply our new concept of total tail dependence to these model mixtures.

The paper is organized as follows. In Section 2, we define the new notion of total tail dependence along diagonals. In Section 3, we apply this concept to the case of model mixtures for the measurement of total tail dependence. In Section 4, we illustrate the new concept of total tail dependence using some empirical examples on stock market indices and exchange rates. In Section 5, we provide an extension to the multivariate case and consider total tail dependence for model mixtures.

2. Total tail dependence

In this section, we propose an approach for modeling total tail dependence. It combines homogeneous tail dependence in the same direction along the principal diagonal and

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heterogeneous tail dependence in the opposite direction along the minor diagonal. This new notion is connected to classical concepts of upper tail and lower tail dependence (Joe, 1997; Jogdeo, 1982).

Let $(X, Y)^\top$ be a random vector with marginal distribution functions F and G, domain $\mathbb{R}^2$ and joint distribution function H. If F and G are continuous, there exists a unique bivariate copula C such that for all $(x, y)^\top \in \mathbb{R}^2$,

$$H(x, y) = C(F(x), G(y)). \tag{1}$$

For applications of the copula concept to quantitative risk, see Embrechts et al. (2003) and Embrechts et al. (1999), among others. So far, the concept of tail dependence has mainly been applied in the upper–upper tail and in the lower–lower tail case. In our paper, we combine them by considering extremal moves off the main diagonal.

In practice, the tail dependent events associated with a random vector $(X, Y)^\top$ do not only appear in the regions $[x, +\infty) \times [y, +\infty)$ (upper–upper tail), $(-\infty, x^*] \times (-\infty, y^*]$ (lower–lower tail). They also occur in the regions $[x, +\infty) \times (-\infty, y^*]$ (upper–lower tail), $(-\infty, x^*] \times [y, +\infty)$ (lower–upper tail). The latter two cases may be referred to as heterogeneous behaviour, as opposed to homogeneous behaviour. In Fig. 1, we illustrate the concept both at the level of the original pair (X, Y), as well as at the level of the underlying copula (U, V). Because we still restrict to extremal events along diagonals of the square $[0, 1]^2$ in the copula space, we refer to these concepts as total tail dependence along the diagonals.

In mathematical form, given a random vector $(X, Y)^\top$ associated with a bivariate copula $C(u, v)$, this total tail dependence along the diagonals is defined in quantile-based by the matrix:

$$\Lambda(q) = \begin{pmatrix} \lambda_{\mathrm{LU}}(q) & \lambda_{\mathrm{UU}}(q) \\ \lambda_{\mathrm{LL}}(q) & \lambda_{\mathrm{UL}}(q) \end{pmatrix}, \tag{2}$$

where

$$\begin{cases} \lambda_{\mathrm{UU}}(q) = \Pr[V > q | U > q] \\ \lambda_{\mathrm{UL}}(q) = \Pr[V < (1-q) | U > q] \\ \lambda_{\mathrm{LL}}(q) = \Pr[V < (1-q) | U < (1-q)] \\ \lambda_{\mathrm{LU}}(q) = \Pr[V > q | U < (1-q)] \end{cases} \tag{3}$$

for all $q \in (0.5, 1.0)$.

The measure of total tail dependence of $(X, Y)^\top$ can also be expressed in asymptotic form as the matrix

$$\Lambda = \lim_{q\to 1^-} \Lambda(q) = \begin{pmatrix} \lim_{q\to 1^-} \lambda_{\mathrm{LU}}(q) & \lim_{q\to 1^-} \lambda_{\mathrm{UU}}(q) \\ \lim_{q\to 1^-} \lambda_{\mathrm{LL}}(q) & \lim_{q\to 1^-} \lambda_{\mathrm{UL}}(q) \end{pmatrix} \tag{4}$$

provided that the limits exist.

Obviously, (4) can also be written as

$$\Lambda = \lim_{q\to 1^-} \begin{pmatrix} \dfrac{1-q-C(1-q,q)}{1-q} & \dfrac{1-2q+C(q,q)}{1-q} \\ \dfrac{C(1-q,1-q)}{1-q} & \dfrac{1-q-C(q,1-q)}{1-q} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lim_{q\to 1^-} \dfrac{C(1-q,q)}{1-q} & \lim_{q\to 1^-} \dfrac{1-C(q,q)}{1-q} \\ -\lim_{q\to 1^-} \dfrac{C(1-q,1-q)}{1-q} & \lim_{q\to 1^-} \dfrac{C(q,1-q)}{1-q} \end{pmatrix}. \tag{5}$$

As is common in the literature, $\lambda_{\mathrm{UU}} > 0$ is takento mean that there exists asymptotic upper–upper tail dependence between the extreme events $\{Y > G^{-1}(q)\}$ and $\{X > F^{-1}(q)\}$. Also, $\lambda_{\mathrm{LL}} > 0$ means that there exists asymptotic lower–lower tail dependence between the extreme events $\{Y < G^{-1}(1-q)\}$ and $\{X < F^{-1}(1-q)\}$.

Similarly, $\lambda_{\mathrm{UL}} > 0$ means that there exists asymptotic upper–lower tail dependence between the extreme events $\{Y < G^{-1}(1-q)\}$ and $(\{X > F^{-1}(q)\})$. Also, $\lambda_{\mathrm{LU}} > 0$ means that there exists asymptotic lower–upper tail dependence between the extreme events $\{Y > G^{-1}(q)\}$ and $\{X < F^{-1}(1-q)\}$.

Moreover, λ_{UU} and λ_{LL} correspond to λ_{U} and λ_{L} (Joe, 1997) and there does not exist any tail dependence of $(X, Y)^\top$ in asymptotic form (or in quantile-based) if $\Lambda = 0$ (or $\Lambda(q) = 0$).

It is straightforward to obtain empirical estimation of the entries in Λ on the base of observations $\Omega = \{(x_i, y_i)^\top\}_{i=1}^n$ as shown in the following matrix:

$$\hat{\Lambda}(q, n) = \begin{pmatrix} \lambda_{\mathrm{LU}}(q, n) & \lambda_{\mathrm{UU}}(q, n) \\ \lambda_{\mathrm{LL}}(q, n) & \lambda_{\mathrm{UL}}(q, n) \end{pmatrix}, \tag{6}$$

where

$$\begin{cases} \lambda_{\mathrm{UU}}(q, n) = \dfrac{1}{n(1-q)} \sum_{i\le n} I_{(x_i > x_{[nq]:n},\, y_i > y_{[nq]:n})} \\ \lambda_{\mathrm{UL}}(q, n) = \dfrac{1}{n(1-q)} \sum_{i\le n} I_{(x_i > x_{[nq]:n},\, y_i < y_{[n(1-q)]:n})} \\ \lambda_{\mathrm{LL}}(q, n) = \dfrac{1}{n(1-q)} \sum_{i\le n} I_{(x_i < x_{[n(1-q)]:n},\, y_i < y_{[n(1-q)]:n})} \\ \lambda_{\mathrm{LU}}(q, n) = \dfrac{1}{n(1-q)} \sum_{i\le n} I_{(x_i < x_{[n(1-q)]:n},\, y_i > y_{[nq]:n})} \end{cases} \tag{7}$$

for all $q \in (0.5, 1.0)$. Here, $I(\cdot)$ is an indicator function and $x_{i:n}$ and $y_{j:n}$ denote the order statistics from the sample $\{(x_i, y_i)^\top\}_{i=1}^n$.

For example, if $(X, Y)^\top$ has a standard bivariate normal distribution with linear correlation ρ, the total tail dependence of the normal copula,

$$C^{N(\rho)}(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \times \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \exp\left\{-\frac{(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\right\} \mathrm{d}s\mathrm{d}t,$$

is described in asymptotic form by the matrix:

$$\Lambda^{N(\rho)} = \begin{pmatrix} \lambda_{\mathrm{LU}}^{N(\rho)} & \lambda_{\mathrm{UU}}^{N(\rho)} \\ \lambda_{\mathrm{LL}}^{N(\rho)} & \lambda_{\mathrm{UL}}^{N(\rho)} \end{pmatrix} = \begin{pmatrix} \delta(1+\rho) & \delta(1-\rho) \\ \delta(1-\rho) & \delta(1+\rho) \end{pmatrix}, \tag{8}$$