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Ruin theory for a Markov regime-switching model under a threshold dividend strategy

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Abstract

In this paper, we study a Markov regime-switching risk model where dividends are paid out according to a certain threshold strategy depending on the underlying Markovian environment process. We are interested in these quantities: ruin probabilities, deficit at ruin and expected ruin time. To study them, we introduce functions involving the deficit at ruin and the indicator of the event that ruin occurs. We show that the above functions and the expectations of the time to ruin as functions of the initial capital satisfy systems of integro-differential equations. Closed form solutions are derived when the underlying Markovian environment process has only two states and the claim size distributions are exponential.

Keywords: Markov regime-switching; Dividend; Ruin probability; Deficit at ruin; Integro-differential equation

1. Introduction

Recently ruin theory under regime-switching model is becoming a popular topic. This model is proposed in Reinhard (1984) and Asmussen (1989). Asmussen calls it a Markov-modulated risk model. The model builds a Markov chain whose states represent different states of an economy into the insurance risk model. The regime switching of the states of the economy can be attributed to the structural changes in the (macro-)economic conditions, the changes in political regimes, the impact of (macro-)economic news and business cycles, etc. There are many papers published on ruin probabilities under the Markov regime-switching model. For example, Baeuerle (1996) investigates the expected ruin time of a Markov-modulated risk model. Lu and Li (2005) study ruin probabilities under this model. Ng and Yang (2005) obtain an upper bound for the joint distribution of surplus before and at ruin under the regime-switching model by using a martingale approach. Ng and Yang (2006) present some explicit results for the joint distribution of surplus before and at ruin under this model in the cases of zero initial surplus and phase type claim size distributions, respectively.

In Gerber (1973, 1979), ruin probability is studied under the classical insurance risk model with a linear dividend barrier. Lin et al. (2003) study the Gerber–Shiu function under an insurance risk model with dividends. Lin and Pavlova (2006) study the Gerber–Shiu function under a classical insurance risk model with a threshold dividend strategy. Readers are referred to Gerber and Shiu (2004) for a set of references for dividend related problems.

In this paper, we assume that the states of the Markov chain can be explicitly observed. Since the premium and the claims process of the model depend on the state of the underlying Markov chain, it is natural to assume that the dividend threshold level and the dividend payment rate also depend on the state of the Markov chain. We consider the regime-switching insurance risk model with a threshold dividend strategy. Those models with a constant dividend threshold level and a constant payment rate are special cases of our model. Our paper can be considered as an extension of the model in Lin and Pavlova (2006) or a modification of the model in Baeuerle (1996).

In this paper, the ruin probability, the deficit at ruin and the expected ruin time are studied. We work with functions that involve the deficit at ruin and the indicator of the event that ruin occurs. We also treat the expectations of the time to ruin as functions of the initial value of the risk process. Systems of integro-differential equations satisfied by these quantities

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are obtained. In the two-state case, closed form solutions are obtained if claim sizes follow exponential distributions.

The paper is organized as follows. The model is presented in Section 2. In Section 3, we show that in the Markov-modulated risk model, there also exists a condition similar to the positive safety loading condition in the classical model, which guarantees that the ruin probability is strictly less than 1. We deal with the functions related to the ruin probability and the deficit at ruin in Section 4. We derive systems of integro-differential equations satisfied by the functions concerned and thereafter transform the equations into second-order ordinary differential equations under the assumption that claim sizes follow exponential distributions. Therefore our problems are reduced to solving systems of ordinary differential equations. Closed form solutions for the concerned quantities in the case that the environment process has two states are obtained. In Section 5, we derive the system of integro-differential equations satisfied by the expected ruin time and obtain the closed form solution in the two-state case with exponential claim size distributions. Numerical examples are given in the last section.

2. The model

Consider a Markov regime-switching risk model where the premium rates, claim inter-arrival times and claim size distributions depend on a stationary Markov process $\{J_t\}_{0\le t<\infty}$ with a finite state space $\mathcal{E} = \{1, 2, \ldots, m\}$, which is governed by the intensity matrix $Q = (q_{ij})_{m\times m}$. At time t, given J_t, the premium rate is p_{J_t}, claims arrive according to a Poisson process with intensity λ_{J_t}, and if a claim arrives at time t the claim size distribution function is $F_{J_t}(\cdot)$. Without confusion, we sometimes use the notation F_i, $i \in \mathcal{E}$, to represent the corresponding distributions and μ_{F_i}, $i \in \mathcal{E}$, to represent the corresponding expectations. We denote by U_n the size of the nth claim and S_n the arrival time of the nth claim. Given $\{J_{S_n}\}_{n\in\mathbb{N}}$, the sequence of claim sizes $U_1, U_2, \ldots$ are assumed to be mutually independent and independent of $\{S_n\}_{n\in\mathbb{N}}$ and $\{J_t\}_{t\ge0}$. We further assume that $\{J_t\}_{t\ge0}$ is time-homogeneous, irreducible and recurrent, and π is its invariant distribution. We use the Markov-modulated Poisson process N_t to denote the number of claims up to time t, i.e. $N_t = \max\{n > 0 : S_n \le t\}$.

Furthermore we assume that given J_t, the insurer pays dividends at rate d_{J_t} to policyholders if the surplus at this time exceeds the level b_{J_t}. The risk surplus process is defined by

$$R_t = R_0 + \int_0^t (p_{J_s} I\{R_{s-} < b_{J_s}\} + (p_{J_s} - d_{J_s}) I\{R_{s-} \ge b_{J_s}\})\mathrm{d}s - \sum_{l=1}^{N_t} U_l, \tag{2.1}$$

where R_0 is a random variable independent of $\{J_t\}$, $\{N_t\}$ and $\{U_n\}$. Throughout the paper, we base our study on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathrm{P})$, where $\mathcal{F}_t = \sigma\{(R_s, J_s) : 0 \le s \le t\}$. Note that the process $\{(R_t, J_t)\}_{t\ge0}$ is a time-homogeneous Markov process.

In the classical risk theory, when we talk about ruin probabilities, we always assume that the net-profit condition holds, otherwise ruin is certain. We will show, in the next section, that there is a similar condition for this dividend modified process to guarantee that ruin is not certain.

Denote the time to ruin by

$$T = \inf\{t \ge 0 : R_t < 0\}$$

and define

$$\mathrm{P}_{(x,i)}(\cdot) = \mathrm{P}(\cdot|R_0 = x, J_0 = i),$$
$$\mathrm{E}_{(x,i)}(\cdot) = \mathrm{E}[\cdot|R_0 = x, J_0 = i].$$

We are interested in the conditional ruin probabilities, $\psi(x; i) = \mathrm{P}_{(x,i)}(T < \infty)$, the distribution of the deficit at ruin, $\phi(x, y; i) = \mathrm{P}_{(x,i)}(T < \infty, |R_T| \le y)$, and the expectation of the time to ruin, $V(x; i) = \mathrm{E}_{(x,i)}[T]$, given the initial surplus x and the initial environment state $i \in \mathcal{E}$. Define, for any fixed i,

$$L(x; i) = \mathrm{E}_{(x,i)}[\mathrm{e}^{-\delta|R_T|} I\{T < \infty\}], \quad \text{for } \delta \ge 0.$$

Then for any fixed $i \in \mathcal{E}$, $\psi(x; i) = L(x; i)|_{\delta=0}$ and $L(x; i)$ is the Laplace–Stieljes transform of $\phi(x, y; i)$ with respect to y with parameter δ, i.e.

$$L(x; i) = \int_0^\infty \mathrm{e}^{-\delta y} \phi(x, dy; i).$$

So it is sufficient for us to investigate the functions $L(x; i)$ and $V(x; i)$ for $i \in \mathcal{E}$. We will show that $L(x; i)$ and $V(x; i)$ for $i \in \mathcal{E}$ satisfy certain systems of integro-differential equations. When the claim size distributions are of exponential type, we obtain second-order ordinary linear differential equations with constant coefficients satisfied by these quantities. The equations are solved when the Markov chain $\{J_t\}_{t\ge0}$ has only two states.

3. The net-profit condition

In this section, we consider the net-profit condition for our model. Our result indicates that the ruin probabilities will be less than 1 no matter what the premium rates are when the surplus falls below the maximal threshold level $\max_{i\in\mathcal{E}}\{b_i\}$ as long as the net-profit condition holds when the surplus is above the maximal dividend level.

Theorem 3.1. *For $x \ge 0$, the ruin probabilities $\psi(x; i) < 1$ for $i \in \mathcal{E}$, if and only if $\sum_{i\in\mathcal{E}} \pi_i \lambda_i \mu_{F_i} < \sum_{i\in\mathcal{E}} \pi_i (p_i - d_i)$.*

In the following proof, we define $\bar{R}_0$ and $\tilde{R}_0$ to be two random variables independent of $\{J_t\}$, $\{N_t\}$ and $\{U_n\}$.

Proof. Assume that $\sum_{i\in\mathcal{E}} \pi_i \lambda_i \mu_{F_i} < \sum_{i\in\mathcal{E}} \pi_i (p_i - d_i)$. Define a risk process

$$\bar{R}_t = \bar{R}_0 + \int_0^t (p_{J_s} - d_{J_s})\mathrm{d}s - \sum_{l=1}^{N_t} U_l,$$

and let $\bar{T}$ be the corresponding time to ruin. Then

$$\mathrm{P}(T < \infty) \le \mathrm{P}(\bar{T} < \infty) < 1.$$

Proof of the last inequality can be found in Reinhard (1984).