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Weighted premium calculation principles

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Abstract

A prominent problem in actuarial science is to define, or describe, premium calculation principles (pcp's) that satisfy certain properties. A frequently used resolution of the problem is achieved via distorting (e.g., lifting) the decumulative distribution function, and then calculating the expectation with respect to it. This leads to coherent pcp's. Not every pcp can be arrived at in this way. Hence, in this paper we suggest and investigate a broad class of pcp's, which we call weighted premiums, that are based on *weighted* loss distributions. Different weight functions lead to different pcp's: any constant weight function leads to the net premium, an exponential weight function leads to the Esscher premium, and an indicator function leads to the conditional tail expectation. We investigate properties of weighted premiums such as ordering (and in particular loading), invariance. In addition, we derive explicit formulas for weighted premiums for several important classes of loss distributions, thus facilitating parametric statistical inference. We also provide hints and references on non-parametric statistical inferential tools in the area.

JEL classification: IM10; IM30; IE43; G32

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1. Introduction and motivation

Let $X \geq 0$ be a loss random variable with cumulative distribution function (cdf) F. The net premium of X, which is the expected value $\mathbf{E}[X]$, can be written in terms of its decumulative distribution function (ddf) $\bar{F} := 1 - F$ as follows:

$$\mathbf{E}[X] = \int_{[0,\infty)} \bar{F}(x)\mathrm{d}x. \tag{1.1}$$

("$:=$" stands for "equality by definition".) Note that we can also rewrite the above integral in terms of the quantile function $F^{-1}(q) := \inf\{x \geq 0 : F(x) \geq q\}$, as the integral $\int_0^1 F^{-1}(t)\mathrm{d}t$. These two ways of writing the mean $\mathbf{E}[X]$ lead to two interesting and fruitful approaches to creating new premium calculation principles (pcp's), as we shall see below.

Premiums are generally required to be at least as large as the net premium $\mathbf{E}[X]$. Otherwise, the insurer loses money on average; in fact, the insurer defaults with probability 1, meaning that the ruin is certain. Premiums satisfying this lower bound $\mathbf{E}[X]$ are called '*loaded*'. A natural way to arrive at loaded premiums is to lift the ddf $\bar{F}$ up, which can be done by using, for example, the power function t^{ρ} with some $\rho \in (0, 1]$. The latter function is usually called the PH ('proportional hazard') distortion function. In general, let $g : [0, 1] \to [0, 1]$ be an increasing function, called 'distortion function', such that $g(0) = 0$ and $g(1) = 1$. Define (see Denneberg (1994), and Wang (1995, 1996))

$$R[g, F] := \int_{[0,\infty)} g(\bar{F}(x))\mathrm{d}x.$$

This gives a large class of premiums, which have been well-explored in the literature and shown to satisfy desirable actuarial properties. In addition to the distortion premium $R[g, F]$, there are of course numerous other pcp's: see, e.g., Tsanakas and Desli (2003), Heilpern (2003), Young (2004), Denuit et al. (2006), Dhaene et al. (2006), and the references therein.

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There are statistical inferential (see, e.g., Jones and Zitikis (2003, 2005), Brazauskas and Kaiser (2004), Jones et al. (2006), Brazauskas et al. (2007) and other reasons to express the premium, or risk measure, $R[g,F]$ as follows:

$$R[g,F]=\int_0^1 F^{-1}(t)g'(1-t)\mathrm{d}t. \tag{1.2}$$

(Of course, we assume here that the function $g(t)$ is differentiable.) Thus, $R[g,F]$ is a *weighted* integral, with a weight function distorting the quantile function $F^{-1}(t)$. Given this interpretations of $R[g,F]$, it now leaps to mind the idea of using weighted distributions, which are well-known and widely used in statistics, to construct premiums. We shall see below that this idea leads to a variety of premiums, and unifies many known ones. We shall also see that these 'weighted' premiums possess a number of desirable actuarial properties.

The rest of the paper is organized as follows. We introduce weighted premiums in Section 2 and derive their closed-form expressions in a number of parametric models. Further examples are provided in Section 3. In Section 4 we introduce 'generalized weighted' premiums and discuss their basic properties. In Section 5 we consider further properties such as scale and translation invariance, additivity-type properties. Several notes of technical nature are relegated to Appendix.

2. Weighted premiums

Let $w:[0,\infty)\to[0,\infty)$ be a function such that the expectation $\mathbf{E}[w(X)]$ is strictly positive and finite. Then the weighted cdf (see, e.g., Patil and Rao (1978), Patil et al. (1986a,b), Rao (1997); and the references therein) is defined by

$$F_w(x):=\frac{\mathbf{E}[\mathbf{1}\{X\le x\}w(X)]}{\mathbf{E}[w(X)]},$$

where, given a statement S, the indicator $\mathbf{1}\{S\}$ is equal to 1 if S is true and 0 otherwise. Imitating Eq. (1.1), we arrive at the 'weighted' premium

$$H[F_w]:=\int_{[0,\infty)}\bar{F}_w(x)\mathrm{d}x. \tag{2.1}$$

A number of known premiums are special cases of $H[F_w]$. To see this, we write the equation (see Heilmann (1989) and Kamps (1998))

$$H[F_w]=\frac{\mathbf{E}[Xw(X)]}{\mathbf{E}[w(X)]}. \tag{2.2}$$

It is now clear that the premium $H[F_w]$ includes (see Remark 1 in Heilmann (1989)):

$$\left.\begin{array}{l}
\bullet\ \text{net premium } \mathbf{E}[X] \text{ when } w(x)\equiv\text{const},\\
\bullet\ \text{modified variance premium } \mathbf{E}[X]+\mathbf{Var}[X]/\mathbf{E}[X]\\
\quad \text{when } w(x)=x,\\
\bullet\ \text{Esscher's premium } \mathbf{E}[X\mathrm{e}^{\lambda X}]/\mathbf{E}[\mathrm{e}^{\lambda X}]\\
\quad \text{when } w(x)=\mathrm{e}^{\lambda x},\\
\bullet\ \text{Kamps's premium } \mathbf{E}[X(1-\mathrm{e}^{-\lambda X})]/\mathbf{E}[1-\mathrm{e}^{-\lambda X}]\\
\quad \text{when } w(x)=1-\mathrm{e}^{-\lambda x},
\end{array}\right\} \tag{2.3}$$

tail-based premiums such as

$$\left.\begin{array}{l}
\bullet\ \text{CTE (conditional tail expectation) } \mathbf{E}[X|X>x_q]\\
\quad \text{when } w(x)=\mathbf{1}\{x>x_q\},\\
\bullet\ \text{TV (modified tail variance) } \mathbf{E}[X|X>x_q]\\
\quad +\mathbf{Var}[X|X>x_q]/\mathbf{E}[X|X>x_q]\\
\quad \text{when } w(x)=x\mathbf{1}\{x>x_q\},
\end{array}\right\} \tag{2.4}$$

where x_q is the quantile $F^{-1}(q)$. (We use x_q and $F^{-1}(q)$ interchangeably, depending on which notation is more convenient; both are standard in the literature.) For properties of the Esscher premium, we refer to, e.g., van Heerwaarden et al. (1989) and the references therein. For details on tail-based pcp's, we refer to Furman and Landsman (2007a,b).

Note that the first two premiums in (2.3) are based on functions $w(x)$ that do not depend on any parameter, whereas the third and fourth ones (i.e., Esscher's and Kamps's) incorporate a parameter λ. The premiums in (2.4) are based on weight functions $w(x)$ that depend on F, which makes the functions unknown in practice. We shall keep this in mind when discussing statistical inferential methods later in the paper.

Since the function $w(x)$ is non-negative, the premium $H[F_w]$ takes on the values in the range of the risk X, as desired:

$$\operatorname{ess\,inf}[X]\le H[F_w]\le \operatorname{ess\,sup}[X]. \tag{2.5}$$

The right-hand bound is known in the literature as the 'no rip-off' condition. Bounds (2.5) also imply the so-called 'no unjustified risk loading' property, which mathematically speaking means that if the loss X is equal (almost surely) to a constant, then the loading $H[F_w]-\mathbf{E}[X]$ is zero. Naturally, the loading should be non-negative, which leads to the 'loading' property $H[F_w]\ge\mathbf{E}[X]$. We shall see from the general results in Section 4 that $H[F_w]$ satisfies the loading property for any non-decreasing function $w(x)$.

To further illustrate weighted distributions and weighted premiums, consider the case when the weight function $w(x)$ is

$$w_c(x):=x^c$$

for some fixed $c\in(0,1]$. In this case, the cdf F_{w_c} is usually called 'size-biased' (see Patil and Ord (1976)). For a list of other weight functions that appear in the literature, we refer to Patil et al. (1986b). Obviously,

$$H[F_{w_c}]=\frac{\mathbf{E}\left[X^{1+c}\right]}{\mathbf{E}\left[X^c\right]}.$$

(Note in passing that $H[F_{w_c}]$ converges to the net premium when $c\downarrow 0$ and to the modified variance premium when $c\uparrow 1$.) Since the weighted premium $H[F_{w_c}]$ is the ratio of two moments, it can be easily calculated for many distributions. Furthermore, a number of well-known distributions are closed with respect to the transformation $w_c(x)$ (see Patil and Ord (1976)), which makes the calculation of $H[F_{w_c}]$ easy, as we shall see in the next three examples.

Example 2.1. Let $X\backsim \mathrm{Ga}(\gamma,\alpha)$, the gamma distribution with parameters $\gamma>0$ and $\alpha>0$. The probability density function