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Ruin probability in the presence of interest earnings and tax payments

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ABSTRACT

In this paper we investigate the ruin probability in a general risk model driven by a compound Poisson process. We derive a formula for the ruin probability from which the Albrecher–Hipp tax identity follows as a corollary. Then we study, as an important special case, the classical risk model with a constant force of interest and loss-carried-forward tax payments. For this case we derive an exact formula for the ruin probability when the claims are exponential and an explicit asymptotic formula when the claims are subexponential.

1. Introduction

In the classical risk model, the surplus process of an insurer is described as

$$U(t) = u + ct - S(t), \quad t \geq 0.$$

Here, $u \geq 0$ is the initial surplus, $c > 0$ is the constant premium rate, and $S(t) = \sum_{i=1}^{N(t)} X_i$ is a compound Poisson process modelling aggregate claims having the Poisson parameter $\lambda > 0$ and individual claim-size distribution F_X with $F_X(0) = 0$ and mean $\mu > 0$. An important quantity in risk theory is the (infinite-time) ruin probability

$$\Psi(u) = \Pr\left(U(t) < 0 \text{ for some } t \geq 0 | U(0) = u\right).$$

Denote by $\Phi(u) = 1 - \Psi(u)$ the non-ruin probability.

Albrecher and Hipp (2007) extended the study to incorporate tax payments. They proposed a loss-carried-forward tax scheme with a constant tax rate $\gamma \in [0, 1)$. That is, tax is paid at a fixed rate $\gamma \in [0, 1)$ whenever the insurer is in a "profitable situation". The reader is referred to their paper for more details about the loss-carried-forward tax scheme. The modified surplus at time t is written as $U_\gamma(t)$ and the corresponding ruin and non-ruin probabilities are denoted by $\Psi_\gamma(u)$ and $\Phi_\gamma(u)$, respectively. Using conditioning techniques and product identities and assuming that the insurer is in a "profitable condition" immediately after time 0, they established the following remarkably simple formula:

$$\Phi_\gamma(u) = [\Phi(u)]^{\frac{1}{1-\gamma}}. \tag{1.1}$$

Subsequently, Albrecher et al. (2009) refined the proof of (1.1) by linking queueing concepts with risk theory and extended the identity to arbitrary surplus-dependent tax rates.

In this paper we are interested in the ruin probability of a general risk model whose surplus process at time t is denoted by $U_g(t)$ and characterized by the following stochastic differential

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equation (SDE):

$$\mathrm{d}U_g(t) = \begin{cases} c_1\left(U_g(t)\right)\mathrm{d}t - \mathrm{d}S(t), & \text{if } U_g(t) < M_g(t), \\ c_2\left(U_g(t)\right)\mathrm{d}t - \mathrm{d}S(t), & \text{if } U_g(t) = M_g(t), \end{cases} \tag{1.2}$$

where $c_1(\cdot)$ and $c_2(\cdot)$ are two positive functions, and $M_g(t) = \max\{U_g(s), 0 \le s \le t\}$ denotes the running maximum of the surplus process. Whenever the surplus is at the running maximum, the company is according to the terminology of Albrecher and Hipp (2007) in a "profitable situation". For initial surplus $u \ge 0$, denote by $\Psi_g(u)$ and $\Phi_g(u)$ the corresponding ruin and non-ruin probabilities, respectively. Note that although the surplus process $\{U_g(t), t \ge 0\}$ does not possess the Markov property, the pair $\{(U_g(t), M_g(t)), t \ge 0\}$ does.

First, we derive a formula for the ruin probability in the general risk model (1.2) and we show that the Albrecher–Hipp tax identity follows as a corollary. Then, we consider an important special case of the risk model (1.2) where $c_1(x) = c + \delta x$ and $c_2(x) = (c+\delta x)(1-\gamma(x))$, with $\delta > 0$ interpreted as a constant force of interest and $\gamma(x) \in [0, 1)$ as a surplus-dependent tax rate. Denote by $\Psi_{\delta,\gamma}(u)$ and $\Phi_{\delta,\gamma}(u)$ the corresponding ruin and non-ruin probabilities, respectively. We shall drop the subscript γ whenever it is zero unless any confusion could be caused.

The rest of this paper is organized as follows: Section 2 studies the behavior of the general risk model (1.2) and extends the Albrecher–Hipp tax identity as a by-product, Section 3 derives an exact formula for $\Phi_{\delta,\gamma}(u)$ for the case of exponentially distributed claims, Section 4 obtains an explicit asymptotic formula for $\Psi_{\delta,\gamma}(u)$ for the case of subexponential claims, and Section 5 tests the accuracy of the asymptotic formula by some numerical examples.

2. General discussion on ruin probability

The central result of this section is Proposition 2.1 given below. From this, the Albrecher–Hipp tax identity follows as a corollary.

2.1. A key formula

For $x \ge u \ge 0$, let $h(u, x)$ denote the probability that the surplus process $\{U_g(t), t \ge 0\}$, having initial value u, will reach the level x before possible ruin. Trivially, $h(x, x) = 1$. Furthermore, we define a function $q(x)$, which is a conditional probability, as follows. Conditioning on that as the surplus process upcrosses the level x for the first time, there is a claim at that instant, $q(x)$ denotes the probability that ruin occurs before the surplus returns to the level x. Thus, $1 - q(x)$ gives the probability that the surplus stays nonnegative before its return to the level x. Note that $q(x)$ depends on the function $c_1(\cdot)$ but not on the function $c_2(\cdot)$. As the model assumes that at time 0 the surplus is at its running maximum, $q(u)$ is well defined.

We have the following:

Proposition 2.1. *Consider the general risk model (1.2). Then, for $u < x$,*

$$h(u, x) = \exp\left\{-\int_u^x \frac{\lambda q(y)}{c_2(y)}\mathrm{d}y\right\}. \tag{2.1}$$

Proof. By considering whether or not there is a claim during the infinitesimal time interval from 0 to $\mathrm{d}t$, we have

$$\begin{aligned} h(u,x) &= (1-\lambda\mathrm{d}t)h(u+c_2(u)\mathrm{d}t, x) + \lambda\mathrm{d}t\cdot(1-q(u))h(u,x) \\ &= h(u+c_2(u)\mathrm{d}t, x) - \lambda\mathrm{d}t\cdot q(u)h(u,x) \\ &\quad - \lambda\mathrm{d}t\left[h(u+c_2(u)\mathrm{d}t, x) - h(u,x)\right], \end{aligned}$$

which leads to the differential equation

$$c_2(u)\frac{\partial h(u,x)}{\partial u} - \lambda q(u)h(u,x) = 0.$$

Formula (2.1) follows from this equation and the boundary condition $h(x, x) = 1$. □

By definition, $h(u, \infty) = \Phi_g(u)$. Therefore, an immediate consequence of Proposition 2.1 is as follows:

Corollary 2.1. *Consider the general risk model (1.2). Then*

$$\Phi_g(u) = \exp\left\{-\int_u^\infty \frac{\lambda q(x)}{c_2(x)}\mathrm{d}x\right\}. \tag{2.2}$$

Note that, in general, the probability $q(u)$ is unknown. Nevertheless, by (2.2) it holds that

$$q(u) = \frac{\Phi_g'(u)}{\Phi_g(u)}\frac{c_2(u)}{\lambda}, \tag{2.3}$$

which shows that the probability $q(u)$ and the non-ruin probability $\Phi_g(u)$ can be determined by each other once $c_2(u)$ is known; see, e.g. (2.8) below.

2.2. Extension of the Albrecher–Hipp tax identity

We shall show that an extended version of the Albrecher–Hipp tax identity (1.1) follows from (2.2). Let $c_2(x) = (1-\gamma)c_1(x)$, with $\gamma \in [0, 1)$ interpreted as a constant tax rate. That is, tax is paid at a fixed rate $\gamma \in [0, 1)$ whenever the insurer is in a "profitable situation". If the corresponding non-ruin probability is denoted by $\Phi_{g,\gamma}(u)$ then we have

$$\Phi_{g,\gamma}(u) = \exp\left\{-\int_u^\infty \frac{\lambda q(x)}{(1-\gamma)c_1(x)}\mathrm{d}x\right\} = \left[\Phi_{g,0}(u)\right]^{\frac{1}{1-\gamma}}. \tag{2.4}$$

Therefore, the Albrecher–Hipp tax identity (1.1) corresponds to (2.4) with $c_1(\cdot)$ being a positive constant c.

2.3. In the presence of interest earnings and tax payments

From now on, we consider an important special case of the general risk model (1.2) where $c_1(x) = c + \delta x$ and $c_2(x) = (c+\delta x)(1-\gamma(x))$, with $\delta > 0$ interpreted as a constant force of interest and $\gamma(x) \in [0, 1)$ as a surplus-dependent tax rate.

It follows from Corollary 2.1 that

$$\Phi_{\delta,\gamma}(u) = \exp\left\{-\int_u^\infty \frac{\lambda q(x)}{(c+\delta x)(1-\gamma(x))}\mathrm{d}x\right\}. \tag{2.5}$$

When $\gamma(x) \equiv 0$, formula (2.5) reduces to

$$\Phi_\delta(u) = \exp\left\{-\int_u^\infty \frac{\lambda q(x)}{c+\delta x}\mathrm{d}x\right\}. \tag{2.6}$$

When $\gamma(x) \equiv \gamma \in [0, 1)$ is a constant, formulas (2.5) and (2.6) immediately imply the Albrecher–Hipp tax identity

$$\Phi_{\delta,\gamma}(u) = \left[\Phi_\delta(u)\right]^{\frac{1}{1-\gamma}}. \tag{2.7}$$

Furthermore, similar to (2.3), it follows from (2.6) that

$$q(u) = \frac{\Phi_\delta'(u)}{\Phi_\delta(u)}\frac{c+\delta u}{\lambda}. \tag{2.8}$$