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Tail bounds for the distribution of the deficit in the renewal risk model

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ABSTRACT

We obtain upper and lower bounds for the tail of the deficit at ruin in the renewal risk model, which are (i) applicable generally; and (ii) based on reliability classifications. We also derive two-side bounds, in the general case where a function satisfies a defective renewal equation, and we apply them to the renewal model, using the function Λ_u introduced by [Psarrakos, G., Politis, K., 2007. A generalisation of the Lundberg condition in the Sparre Andersen model and some applications (submitted for publication)]. Finally, we construct an upper bound for the integrated function $\int_y^\infty \Lambda_u(z)\,dz$ and an asymptotic result when the adjustment coefficient exists.

1. Introduction

We consider the general renewal risk model, often referred to as the Sparre Andersen risk model. In this model, the insurer's surplus at time t, which is denoted by $U(t)$, is given by

$$U(t) = u + ct - \sum_{k=1}^{N_t} Y_k, \tag{1}$$

where $u \geq 0$ is the initial surplus, c is the rate of premium income per unit time and N_t is the number of claims in the time interval $(0, t]$. The individual claim amounts $Y_1, Y_2, \ldots$ are positive, independent and identically distributed (i.i.d.) random variables with common distribution function (d.f.) $P(y) = \Pr(Y \leq y)$, tail $\overline{P}(y) = 1 - P(y) = \Pr(Y > y)$, density $p(y)$ and mean $\mathrm{E}(Y_1) < \infty$. These claim amounts are also independent of N_t. We assume that a claim has taken place at time 0, with u being the surplus immediately after this claim has been paid (see Gerber and Shiu (2005)). The corresponding interclaim times $T_1, T_2, \ldots$ are arbitrary i.i.d. positive random variables with common mean $\mathrm{E}(T_1)$. We assume that $c = (1+\theta)\,\mathrm{E}(Y_1)/\mathrm{E}(T_1)$, where $\theta > 0$ is known as the relative safety loading.

The probability of ultimate ruin is defined by

$$\psi(u) = \Pr\left(\inf_{t>0} U(t) < 0 | U(0) = u\right), \quad u \geq 0. \tag{2}$$

The time of ruin is

$$T = \begin{cases} \infty, & \text{if } U(t) \geq 0 \text{ for all } t > 0 \\ \inf\{t > 0 | U(t) < 0\}, & \text{otherwise,} \end{cases}$$

and the probability of ruin is written as

$$\psi(u) = \Pr(T < \infty | U(0) = u).$$

In general, we assume that $\mathrm{E}(Y_1) < c\,\mathrm{E}(T_1)$, so that ruin is not certain to occur.

The distribution of the deficit, namely,

$$H(u, y) = \Pr(|U(T)| \leq y, T < \infty | U(0) = u),$$

was introduced by Gerber et al. (1987) and represents the probability that, starting with a surplus u, ruin occurs and the deficit $|U(T)|$ at the time of ruin T does not exceed $y \geq 0$. It is a defective d.f. with tail

$$\overline{H}(u, y) = \psi(u) - H(u, y) = \Pr(|U(T)| > y, T < \infty | U(0) = u),$$

and satisfies $\lim_{y\to\infty} H(u, y) = \overline{H}(u, 0) = \psi(u) < 1$. It is also convenient to define the proper d.f. of the deficit,

$$H_u(y) = \frac{H(u, y)}{\psi(u)} = \Pr(|U(T)| \leq y \,|\, T < \infty, U(0) = u) \tag{3}$$

with tail $\overline{H}_u(y) = 1 - H_u(y)$.

The main purpose of this article is to obtain improved bounds for the tail of the deficit, $\overline{H}(u, y)$. Recently, Willmot (2002) and Chadjiconstantinidis and Politis (2007) gave bounds for this

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function. In Section 3, we construct also lower and upper bounds for $\overline{H}(u,y)$. For this study, we need some reliability classifications, which are briefly reviewed in the next section. In Section 4, we derive two-sided bounds for a function satisfying a defective renewal equation, improving and generalising a result obtained by Willmot and Lin (2001). The key for our analysis is the function $\Lambda_u(y) = \phi[\psi(u+y) - \psi(u)\psi(y)]/(1-\phi)$, where $\phi = \psi(0)$, introduced by Psarrakos and Politis (2007). Finally, in Section 5, we obtain an upper bound for the integrated function $\int_y^\infty \Lambda_u(z)\,\mathrm{d}z$ and derive an asymptotic result in the case where the adjustment coefficient exists.

2. Definitions and preliminaries

Let $S = \sum_{i=1}^{N} X_i$ denote a compound geometric random variable, where $\Pr(N = n) = (1-\phi)\phi^n$ for $n = 0, 1, 2, \ldots$, and $0 < \phi < 1$. Suppose also that $X_1, X_2, \ldots$ are i.i.d. with ladder height d.f. F and density f. Then we have that $\Pr(S > u) = \psi(u)$, see Bowers et al. (1986, Chapter 12). Furthermore, if $\mathrm{E}(X_1) = \mu < \infty$, then $\mathrm{E}(S) = \mathrm{E}(X_1)\,\mathrm{E}(N) = \mu\phi/(1-\phi)$.

One expression of the probability of ruin is the formula of Pollaczeck and Khinchine, see Asmussen (2000),

$$\psi(u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n\overline{F^{*n}}(u), \tag{4}$$

where $\phi = \psi(0)$. The probability of ruin also satisfies the defective renewal equation, see Willmot and Lin (2001),

$$\psi(u) = \phi\int_0^u \psi(u-z)\,\mathrm{d}F(z) + \phi\overline{F}(u). \tag{5}$$

The solution of this equation is

$$\psi(u) = \frac{\phi}{1-\phi}\int_{0+}^{u}\overline{F}(u-z)\,\mathrm{d}H(z) + \phi\overline{F}(u), \tag{6}$$

where

$$H(u) = \sum_{n=0}^{\infty}(1-\phi)\phi^n F^{*n}(u) \tag{7}$$

is the probability of non-ruin, and

$$\overline{H}(u) = 1 - H(u) = \sum_{n=1}^{\infty}(1-\phi)\phi^n\overline{F^{*n}}(u) = \psi(u).$$

By Willmot (2002), we know that $\overline{H}(u,y)$ satisfies the defective renewal equation

$$\overline{H}(u,y) = \phi\int_0^u \overline{H}(u-t,y)\,\mathrm{d}F(t) + \phi\overline{F}(u+y), \tag{8}$$

whose solution is

$$\overline{H}(u,y) = \frac{\phi}{1-\phi}\int_{0+}^{u}\overline{F}(u+y-t)\,\mathrm{d}H(t) + \phi\overline{F}(u+y). \tag{9}$$

One can easily see that for $y = 0$, the relations (8) and (9) yield (5) and (6), respectively.

A d.f. $A(x)$, $x \geq 0$, with tail $\overline{A}(x) = 1 - A(x)$ is said to be decreasing (increasing) failure rate or DFR (IFR) if $\overline{A}(x+y)/\overline{A}(x)$ is nondecreasing (nonincreasing) in x for any $y \geq 0$. If $A(x)$ is absolutely continuous with density $a(x)$, then it is DFR (IFR) when the failure rate $\lambda_A(x) = a(x)/\overline{A}(x)$ is nonincreasing (nondecreasing). The exponential is the only distribution that is both DFR and IFR, and a mixture of exponential d.f. is always DFR, see Willmot and Lin (2001, Chapter 2).

A d.f. $A(x)$ is called new worse (better) than used or NWU (NBU) if $\overline{A}(x+y) \geq (\leq)\overline{A}(x)\overline{A}(y)$ for every $x, y \geq 0$. The DFR (IFR) class of distributions is a subclass of the NWU (NBU) class.

3. General bounds

In the context of the renewal risk model, an upper bound for the tail of the deficit at ruin, $\overline{H}(u,y)$, in terms of the probability of ruin was given by Willmot (2002, Theorem3.2), who proved that

$$\overline{H}(u,y) \leq \frac{1}{1-\phi}[\psi(u+y) - \psi(u)\psi(y)]. \tag{10}$$

Chadjiconstantinidis and Politis (2007), using the upper bound in (10), derive a better upper bound, that is

$$\overline{H}(u,y) \leq \frac{1}{1-\phi}[\psi(u+y) - \psi(u)\,\psi(y)] - \frac{\phi}{(1-\phi)^2}[\phi - \psi(y)][1-\psi(u)]\overline{F}(u+y). \tag{11}$$

In the following theorem, we give an upper bound for $\overline{H}(u,y)$ that is tighter than the bound (11). It is also worth mentioning that the new bound has a simple form similar to (10).

Theorem 3.1. *For any $u, y \geq 0$, it holds that*

$$\overline{H}(u,y) \leq \frac{1}{1-\psi(y)}[\psi(u+y) - \psi(u)\,\psi(y)]. \tag{12}$$

This bound is always better than the bound in (11).

Proof. By Willmot (2002, relation (2.12)), for any $u, y \geq 0$, we have

$$\psi(u+y) - \psi(u)\psi(y) = \int_{0-}^{y}\overline{H}(u,y-t)\,\mathrm{d}H(t). \tag{13}$$

Since the function $\overline{H}(u,y-t)$ is nondecreasing in $t \in [0,y]$, (13) yields

$$\psi(u+y) - \psi(u)\psi(y) \geq \int_{0-}^{y}\overline{H}(u,y)\,\mathrm{d}H(t) = \overline{H}(u,y)[1-\psi(y)],$$

and after a little rearrangement, the upper bound in (12) follows.

Next we verify that bound (12) is always tighter than bound (11). By Proposition 2.2 of Chadjiconstantinidis and Politis (2007) and (12), we have

$$\frac{\phi}{1-\phi}[1-\psi(u)]\overline{F}(u+y) \leq \frac{1}{1-\psi(y)}[\psi(u+y) - \psi(u)\psi(y)].$$

Multiplying both sides of this inequality by $[\phi - \psi(y)]/(1-\phi)$, it follows

$$\frac{\phi}{(1-\phi)^2}[\phi-\psi(y)][1-\psi(u)]\overline{F}(u+y) \leq \frac{\phi-\psi(y)}{(1-\phi)(1-\psi(y))}[\psi(u+y) - \psi(u)\psi(y)],$$

or equivalently,

$$\frac{\phi}{(1-\phi)^2}[\phi-\psi(y)][1-\psi(u)]\overline{F}(u+y) \leq \left[\frac{1}{1-\phi} - \frac{1}{1-\psi(y)}\right][\psi(u+y) - \psi(u)\psi(y)].$$

It is straightforward to see that

$$\frac{1}{1-\psi(y)}[\psi(u+y) - \psi(u)\psi(y)] \leq \frac{1}{1-\phi}[\psi(u+y) - \psi(u)\psi(y)] - \frac{\phi}{(1-\phi)^2}[\phi-\psi(y)][1-\psi(u)]\overline{F}(u+y),$$

and the proof is complete. $\square$