Insurance: Mathematics and Economics 46 (2010) 32–41

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Surplus analysis for a class of Coxian interclaim time distributions with applications to mixed Erlang claim amounts

Gordon E. Willmot, Jae-Kyung Woo *

Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, Canada

ARTICLE INFO

Article history:
Received January 2009
Received in revised form
July 2009
Accepted 18 August 2009

Keywords:
Sparre Andersen risk process
K_n family of distributions
Combination of Erlangs
Mixtures of Erlangs
Defective renewal equation
Compound geometric distribution
Ladder height
Generalized Lundberg's fundamental equation
Lagrange polynomials

ABSTRACT

Gerber–Shiu analysis with the generalized penalty function proposed by Cheung et al. (in press-a) is considered in the Sparre Andersen risk model with a K_n family distribution for the interclaim time. A defective renewal equation and its solution for the present Gerber–Shiu function are derived, and their forms are natural for analysis which jointly involves the time of ruin and the surplus immediately prior to ruin. The results are then used to find explicit expressions for various defective joint and marginal densities, including those involving the claim causing ruin and the last interclaim time before ruin. The case with mixed Erlang claim amounts is considered in some detail.

1. Introduction and preliminaries

Consider the insurer's surplus process at time t defined as $\{U_t; t \geq 0\}$ with $U_t = u + ct - \sum_{i=1}^{N_t} Y_i$, and $u \geq 0$ is the initial surplus. The number of claims process $\{N_t; t \geq 0\}$ is assumed to be a renewal process, with V_1 the time of the first claim and V_i the time between the $(i-1)$th and the ith claim for $i = 2, 3, 4, \ldots$. It is assumed that $\{V_i\}_{i=1}^{\infty}$ is an independent and identically distributed (iid) sequence of positive random variables with common probability density function (pdf) $k(t)$ and distribution function (df) $K(t) = 1 - \overline{K}(t)$.

In the present paper, we consider the model of Li and Garrido (2005), whereby $k(t)$ is a pdf from the K_n class of densities, and has Laplace transform $\tilde{k}(s) = \int_0^{\infty} e^{-st} k(t) dt$ given by

$$\tilde{k}(s) = \frac{\zeta(s)}{\prod_{i=1}^{m} (\lambda_i + s)^{n_i}} \quad (1)$$

where $\lambda_i > 0$ for $i = 1, 2, \ldots, m$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. Also, n_i is a nonnegative integer for $i = 1, 2, \ldots, m$, and $n = n_1 + \cdots + n_m > 0$, while $\zeta(s)$ is a polynomial of degree $n-1$ or less (the denominator of (1) is a polynomial of degree n). In this paper we adopt the notational convention that the empty product is 1, and the empty sum is 0. The classical compound Poisson risk model (e.g. Gerber and Shiu, 1998) is recovered in the exponential case with $m = n = 1$, the Erlang(n) renewal risk model (Li and Garrido, 2004) with $m = 1$, and $n_m = n$, and the generalized Erlang renewal risk model (Gerber and Shiu, 2005) with $n_i = 1$ for $i = 1, 2, \ldots, n$, and $\zeta(s) = \prod_{i=1}^{m} \lambda_i^{n_i}$ in these cases. As pointed out by Li and Garrido (2005), a partial fraction expression of (1) results in

$$\tilde{k}(s) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(\lambda_i + s)^j} \quad (2)$$

where

$$a_{i,j} = \frac{1}{(n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \left\{ \prod_{k=1, k\neq i}^{m} \frac{\zeta(s)}{(\lambda_k + s)^{n_k}} \right\} \Bigg|_{s=-\lambda_i}.$$

Inversion of (2) results in

$$k(t) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{i,j} \frac{t^{j-1} e^{-\lambda_i t}}{(j-1)!}, \quad (3)$$

and the K_n class may be viewed in terms of finite combinations of Erlangs. Also, it is assumed that the claim sizes $\{Y_i\}_{i=1}^{\infty}$ with Y_i the

* Corresponding author. Tel.: +1 519 888 4567.
E-mail address: jkwoo@math.uwaterloo.ca (J.-K. Woo).

0167-6687/$ – see front matter
doi:10.1016/j.insmatheco.2009.08.004

size of the ith claim are iid positive random variables with pdf $p(y)$, df $P(y) = 1 - \overline{P}(y)$, and Laplace transform $\widetilde{p}(s) = \int_0^\infty e^{-sy}p(y)dy$. Premiums are paid continuously at rate c, and the positive security loading condition $E[Y_1] < cE[V_1]$ is assumed to hold.

Let T be the time to ruin defined by $T = \inf\{t \geq 0 : U(t) < 0\}$ with $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$, and $\delta \geq 0$ may be viewed as a discount factor. The classical Gerber–Shiu discounted penalty function is defined (Gerber and Shiu, 1998) by

$$m_{\delta,12}(u) = E\left[e^{-\delta T} w_{12}(U_{T-}, |U_T|) I(T < \infty) | U_0 = u\right],$$
$$u \geq 0, \tag{4}$$

where $w_{12}(x, y)$ is the so-called penalty function for $x > 0, y > 0$, and $I(.)$ is the indicator function. If ruin occurs, the surplus prior to ruin is U_{T-} and the deficit at ruin is $|U_T|$.

In this paper, we study a generalized form of the Gerber–Shiu discounted penalty function in (4) which includes a new quantity in the penalty function introduced by Cheung et al. (in press-a), i.e.

$$m_\delta(u) = E\left[e^{-\delta T} w\left(U_{T-}, |U_T|, R_{N_T-1}\right) I(T < \infty) | U_0 = u\right],$$
$$u \geq 0, \tag{5}$$

where $R_n = u + \sum_{i=1}^n (cV_i - Y_i)$ for $n = 1, 2, \ldots$, and $R_0 = u$. The discrete process $\{R_n; n = 0, 1, 2, \ldots\}$ thus represents the surplus immediately after claims occur, i.e. R_n is the surplus after the nth claim for $n > 0$, and R_0 is defined to be the initial surplus. Consequently, R_{N_T-1} is the surplus immediately after the second last claim before ruin occurs if $N_T > 1$, and is equal to the initial surplus u if ruin occurs on the first claim. We may study quantities involving R_{N_T-1} such as the last interclaim time before ruin $V_{N_T} = (U_{T-} - R_{N_T-1})/c$, discussed by Cheung et al. (in press-a) in the classical compound Poisson risk model.

There has been a variety of recent papers which include additional variables (beyond the traditional surplus prior to ruin U_{T-} and the deficit at ruin $|U_T|$) in the penalty function. As mentioned, Cheung et al. (in press-a) consider the penalty function (5) in the classical compound Poisson risk model. Cheung et al. (in press-b) discuss underlying mathematical structural properties of even more general Gerber–Shiu functions where the penalty function includes the minimum surplus level before ruin $X_T = \inf_{0\leq t<T} U_t$ as well, but in the more general dependent Sparre Andersen risk process. The results of Cheung et al. (in press-b) thus apply to the model considered in this paper as well, yielding a variety of mathematical properties, but appear to be of less use for explicit identification of some quantities of interest under the present K_n interclaim time assumption. The same variable X_T has also been included in the penalty function in Gerber–Shiu analysis of Levy risk processes in Biffis and Kyprianou (in press) and Biffis and Morales (2009).

Obviously, with $w(x, y, v) = 1$ in (5),

$$\overline{G}_\delta(u) = E\left[e^{-\delta T} I(T < \infty) | U_0 = u\right], \quad u \geq 0, \tag{6}$$

and for $\delta = 0$ (6) is the ruin probability $\psi(u) = \Pr(T < \infty | U_0 = u)$. We remark that $\overline{G}_\delta(u) = 1 - G_\delta(u)$ is a compound geometric tail and satisfies the defective renewal equation

$$\overline{G}_\delta(u) = \phi_\delta \int_0^u \overline{G}_\delta(u-y) f_\delta(y)dy + \phi_\delta \int_u^\infty f_\delta(y)dy, \tag{7}$$

and Li and Garrido (2005) have identified ϕ_δ and the ladder height pdf $f_\delta(y)$.

As was the case in Li and Garrido (2005), the analysis of (5) depends heavily on the Dickson-Hipp operator (e.g. Li and Garrido, 2004) defined for a function $h(x)$ by $T_r h(x) = e^{rx}\int_x^\infty e^{-ry}h(y)dy$. If $\widetilde{h}(s) = \int_0^\infty e^{-sy}h(y)dy$ is the Laplace transform of $h(x)$, the Laplace transform of $T_r h(x)$ is given by

$$\int_0^\infty e^{-sx}\{T_r h(x)\}dx = \{\widetilde{h}(r) - \widetilde{h}(s)\}/(s-r).$$

Also, Lundberg's (generalized) fundamental equation

$$\widetilde{p}(s)\widetilde{k}(\delta - cs) = 1 \tag{8}$$

is of central importance in the ensuing analysis, and Li and Garrido (2005) showed that (8) has exactly n roots $\rho_1, \rho_2, \ldots, \rho_n$ with nonnegative real part $\mathrm{Re}(\rho_j) \geq 0$ in the complex plane. We shall henceforth assume (as did Li and Garrido, 2005) that these roots are distinct, i.e. $\rho_i \neq \rho_j$ for $i \neq j$.

It follows from (1) and (2) that

$$\zeta(\delta - cs) = \left\{\prod_{k=1}^m (\lambda_k + \delta - cs)^{n_k}\right\} \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{a_{i,j}}{(\lambda_i + \delta - cs)^j},$$

is still a polynomial in s of degree $n - 1$ or less. More generally, if $\theta_{i,j}$ are constants, then as pointed out by Li and Garrido (2005),

$$q(s) = \left\{\prod_{k=1}^m (\lambda_k + \delta - cs)^{n_k}\right\} \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{\theta_{i,j}}{(\lambda_i + \delta - cs)^j}, \tag{9}$$

is a polynomial in s of degree $n - 1$ or less. Therefore, from the theory of Lagrange polynomials, (9) may be reexpressed as

$$q(s) = \sum_{i=1}^n q(\rho_i) \left\{\prod_{j=1, j\neq i}^n \frac{s - \rho_j}{\rho_i - \rho_j}\right\}. \tag{10}$$

Next, we note that functions of the form

$$h(u) = \int_0^\infty e^{-\delta t} r(u + ct)k(t)dt, \tag{11}$$

where $r(x)$ is a function and $k(t)$ is given by (3), are of interest in later sections. The Laplace transform of (11) is

$$\begin{aligned}\widetilde{h}(s) &= \int_0^\infty e^{-su} \int_0^\infty e^{-\delta t} r(u+ct)k(t)dtdu \\ &= \int_0^\infty e^{-(\delta - cs)t} \left\{\int_0^\infty e^{-s(u+ct)} r(u+ct)du\right\} k(t)dt \\ &= \int_0^\infty e^{-(\delta - cs)t} \left\{\int_0^\infty e^{-sx} r(x)dx - \int_0^{ct} e^{-sx} r(x)dx\right\} k(t)dt.\end{aligned}$$

Thus if the Laplace transform of $r(x)$ is $\widetilde{r}(s) = \int_0^\infty e^{-sx} r(x)dx$, then

$$\widetilde{h}(s) = \widetilde{r}(s)\widetilde{k}(\delta - cs) - \int_0^\infty e^{-sx} \left\{\int_{x/c}^\infty e^{-(\delta - cs)t} k(t)dt\right\} r(x)dx.$$

It is straightforward but tedious to show using (3) that the Laplace transform of (11) may then be expressed as

$$\widetilde{h}(s) = \widetilde{r}(s)\widetilde{k}(\delta - cs) - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{\theta_{i,j}}{(\lambda_i + \delta - cs)^j}, \tag{12}$$

where

$$\theta_{i,j} = \sum_{k=j}^{n_i} \frac{(-1)^{k-j} a_{i,k} \widetilde{r}^{(k-j)}\left(\frac{\lambda_i + \delta}{c}\right)}{(k-j)! c^{k-j}},$$

and $\widetilde{r}^{(j)}(s) = \int_0^\infty (-x)^j e^{-sx} r(x)dx$. Clearly, (12) is completely specified by $\widetilde{k}(s)$ and $\widetilde{r}(s)$. This Laplace transform relationship is used in Section 2 to derive a defective renewal equation for (5), and to show that this is a generalization of that obtained by Li and Garrido (2005) for its special case (4). This generalization, in addition to allowing for analysis involving R_{N_T-1} and related quantities, is in fact also more natural for use in joint analysis involving T and U_{T-}, as is also discussed.

In Section 3, the results of Section 2 are used to obtained the trivariate "discounted" defective distribution of U_{T-}, $|U_T|$, and