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Time consistency conditions for acceptability measures, with an application to Tail Value at Risk

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Abstract

An acceptability measure is a number that summarizes information on monetary outcomes of a given position in various scenarios, and that, depending on context, may be interpreted as a capital requirement or as a price. In a multiperiod setting, it is reasonable to require that an acceptability measure should satisfy certain conditions of time consistency. Various notions of time consistency may be considered. Within the framework of coherent risk measures as proposed by Artzner et al. [Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. Math. Fin. 9, 203–228], we establish implication relations between a number of different notions, and we determine how each notion of time consistency is expressed through properties of a representing set of test measures. We propose modifications of the standard Tail-Value-at-Risk measure that have stronger consistency properties than the original.

Keywords: Coherent risk measures; Acceptability measures; Nonlinear expectations; Capital requirements; Incomplete markets; Time consistency

1. Introduction

Consider a position[1] giving rise to monetary outcomes that may be different in different scenarios. After a set of scenarios has been constructed and outcomes in each scenario have been determined, one frequently needs to summarize all of this information in a single number (an "acceptability measure") which, depending on the particular application, may for instance be interpreted as a capital requirement or as a price. To determine which acceptability measure is appropriate for a given situation, one may make use of axiom systems and representation theorems. The specification of axioms helps to clarify differences between various acceptability measures that may be proposed.

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[1] We use the term "position" here to refer to any project or contract subject to evaluation on the basis of positive or negative future payoffs. A position may for instance consist of a derivative contract, a portfolio of insurance policies, or the collection of assets and liabilities of a pension fund.

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Given a collection of axioms, it is sometimes possible to prove a representation theorem providing a concrete form for all measures that satisfy the given axioms.

The theory of acceptability measures has roots in statistical decision theory (von Neumann and Morgenstern, 1944; Savage, 1954) and in risk theory (Bühlmann, 1970; Gerber, 1979; Goovaerts et al., 1984). A new impulse in the discussion of axiom systems for acceptability measures arose when capital adequacy rules defined in the Basel Accord and elsewhere were investigated from an axiomatic point of view by Artzner et al. (1999). Their proposed set of axioms defining "coherent" acceptability measures[2] is different from the sets of axioms traditionally used in statistical decision theory and admits a relatively simple representation theorem, which in fact was already proved in a different context by Huber (1981). Acceptability measures relate to the theory of asset pricing for instance via the recent literature on "good-deal" pricing in incomplete markets. From a mathematical point of view, acceptability measures may be viewed as nonlinear expectations, which brings in a connection to yet another strand of research. Whether or not the "coherent" axiom system is suitable in a specific context is of course a matter of debate, and given the wide range of potential applications the investigation of alternatives is certainly warranted. For an entry to the related literature and critical discussion, see for instance Carr et al. (2001), Černý and Hodges (2002), Föllmer and Schied (2004), Frittelli and Rosazza Gianin (2002), Jaschke and Küchler (2001), Wang et al. (1997), Kaas et al. (2003), Landsman and Sherris (2001), Dhaene et al. (2003), Dhaene et al. (2004), and Peng (1997).

The work of Artzner et al. (1999) is formulated in a single-period setting. In many applications, though, it may be envisaged that acceptability measures for a given position will be computed at more than one moment in time, and that acceptability may be enhanced by the use of dynamic strategies. Considerable effort has recently been spent on the development of axiom systems and representation results that apply to dynamic settings (Artzner et al., 2004; Epstein and Schneider, 2003; Frittelli and Scandolo, 2004; Wang, 2003; Riedel, 2004; Roorda et al., 2005; Tsanakas, 2004; Burgert, 2005; Weber, 2006; Tutsch, 2006). In the extension to the dynamic case, a key role is played by notions of *time consistency*. It is the purpose of the present paper to establish the relations between a number of different axiomatizations of the idea of time consistency, working in a finite context (for simplicity) and within the setting of coherent acceptability measures. Straightforward multiperiod extensions of single-period coherent acceptability measures may well give rise to consistency problems, as noted by Artzner (2002) (see also Artzner et al. (2004)) for the case of the Tail-Value-at-Risk measure. We discuss adaptations of this measure which enjoy better consistency properties.

The paper distinguishes essentially three different notions of time consistency, which under mild assumptions are ordered by implication relations. The weakest notion is called *conditional consistency*. This notion requires that the evaluation of a conditional version of a given payoff should be in line with the conditional evaluation of that payoff. The second notion we consider is called *sequential consistency*. This form of time consistency requires that a position cannot be evaluated positively if all conditional evaluations at later stages are negative. The third and strongest notion is called *dynamic consistency*. This notion has been used extensively in the recent work on dynamic risk measures. It is shown here to be closely related to what might be called the *tower law of conditional evaluations*, which holds for a given acceptability measure if evaluations under this measure do not change when the payoffs following a given future event are replaced by their evaluation conditional on that event.

The paper is organized as follows. In Section 2 we summarize the framework. Section 3 introduces the notions of time consistency that we discuss in this paper and that were already mentioned above. Characterizations of these concepts in terms of single-step properties are provided in Section 4. In Section 5 we establish implication relations between different notions of consistency, and we show by examples that the notions we introduce are indeed not equivalent. A major theme in the theory of coherent acceptability measures is representation by means of collections of probability measures. Various representation forms of this type for multiperiod acceptability measures are discussed in Section 6. Subsequently, in Section 7, relations are established between consistency notions and representation properties. Applications to TailVaR are discussed in Section 8, and conclusions follow in Section 9. Most of the proofs of the results of this paper have been collected in Appendix A.

Throughout the paper, we assume that payoffs are represented in terms of a suitable numéraire so that effectively we may suppose that interest rates are zero. This assumption is commonly made in the literature on dynamic acceptability

[2] The cited paper actually employs an opposite sign convention to the one used here and the term "risk measure" is used rather than "acceptability measure". In later work by Artzner and co-authors (Artzner et al., 2004), the same sign convention is used as we do in this paper, and the term "risk-adjusted value" is used for what we call "acceptability measure".